Mathematics and Fiction — illustrated by Grothendieck's contributions to K-theory

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GROTHENDIECK, A MULTIFARIOUS GIANT: MATHEMATICS, LOGIC AND PHILOSOPHY

Chapman University

Mathematics and Fiction — Motivation and Content

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1. Mathematics and fiction: similarities
2. A few general differences between mathematics and fiction
3. Main difference: The role of relations and interconnections in mathematics, illustrated by the early development of $K-$theory
4. Concluding remarks

Main points of paper

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Relations play a more fundamental role in mathematics; not only relations that are internal to a particular structure, but on a more global scale:

1. The mathematical universe is tied together in multiple ways.
2. Relations offers a way to state why mathematics is real: Mathematics is pragmatically real, because it has ties to the physical reality — it can tell us something about the physical world — not because we can formulate an internally consistent story about certain postulated objects.

Characteristics of Mathematics

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- *Relations*: It is common today to say that contemporary mathematics is the study of relations between objects rather than the objects themselves, that is, it studies structures.
- *Global relations*: Mathematical activities (reasoning or introducing new objects), do not only rely on relations of (or relations that define) the considered structure; equally important — as I will show — are the relations being set up between different structures (individual structures or fields of study).

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- They both rely on the same human ability: we create objects and tell stories about them.

- Describing relations between persons, things, places, etc. is as important to a story as it is in mathematics. The type of relations considered, though, are for the most part different and the level of abstraction may differ. In fiction the relations are *concrete*, for example, 'descendant of' or 'live in' whereas in mathematics, they could be transitive or asymmetric relations.

Mathematics and Fiction: differences

I focus on the multiple roles that relations play in maths.

Other differences include (R. Thomas 2000, 2002):

- The aim of the discourse: one of the activities of mathematics concern *reasoning* and so mathematics formulates hypotheses and derive their consequences.
- The clearness and precision of mathematical concepts.
- The (supposed) completeness of mathematical theories, the claim that all the questions that a given context allows us to formulate have definite answers.
- Conversely, mathematical contexts also forbid that certain things can hold because of internal consistence and coherence.
- In fiction, one can freely chose the beginning and ending — not so in mathematics.

K−theory — a brief history

Key events

1857 Riemann formulates a theorem on the existence of complex-valued functions with poles at a number of given points on a Riemann surface with genus g.

1865 Riemann-Roch theorem for curves (how many different such functions is possible to define).

1956 Hirzebruch's Riemann-Roch for algebraic manifolds and vector bundles (topology).

1957 Grothendieck's Riemann-Roch for sheaves over algebraic varieties (algebraic geometry), introducing a K-group.

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Alexander Grothendieck (1928–2014)

Some background for his version of Riemann-Roch

- Became interested in algebraic geometry around 1954-5.
- Tôhoku paper. Homology with coefficients in a sheaf — and abelian category.
- Hirzebruch's version of Riemann-Roch (1956)
- Extreme generality — Bourbaki, category theory

Grothendieck's theorem of Riemann-Roch is published in *Le Théorème de Riemann-Roch* written by A. Borel and J-P. Serre. The article is based on a seminar held at Princeton 1957 presenting Grothendieck's work. (The result is also published in SGA 6)

Hirzebruch-Riemann-Roch

Let V be an n-dimensional algebraic manifold and let W be a complex analytic vector bundle over V with fibre C_q. ... The cohomology groups $H^i(V, W)$ are finite dimensional vector spaces which vanish for $i > n$. The Euler-Poincaré characteristic,

$$\chi(V, W) = \sum_{i=0}^{n} (-1)^i \dim H^i(V, W),$$

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In the following we consider the expression of the Euler-Poincaré characteristic.

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(Recall that V denotes an algebraic manifold and W a vector bundle defined over it.)

We wish to define a homomorphism, $f_!$, from sheaves over X to sheaves over Y that reduces to this expression.

Introducing the $K-$group

Given a map $f : X \to Y$ between two algebraic varieties and a coherent sheaf, $\mathcal{F}$, defined over X it is possible to construct a finite sequence of sheaves over Y that correspond to the cohomology groups, $H^i(V, W)$, the so-called higher direct images of $\mathcal{F}$, denoted by $R^i f_*(\mathcal{F})$.

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First a pre-sheaf over Y is constructed; for each open $U \subset Y$ associate the section $\Gamma(f^{-1}(U), \mathcal{F})$. The resulting sheaf over Y is the direct image.

For each short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ there exists a sequence of sheaves

$0 \to R^0 f_*(\mathcal{F}') \to R^0 f_*(\mathcal{F}) \to R^0 f_*(\mathcal{F}'') \to R^1 f_*(\mathcal{F}') \to R^1 f_*(\mathcal{F}) \to \ldots$

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The obvious expression of the function $f_!$ from sheaves over X to sheaves over Y that reduces to the Euler-Poincaré characteristic then is:

$$f_!(\mathcal{F}) = \sum_{i=0} (-1)^i R^i f_*(\mathcal{F})$$

Introducing the $K-$group

To fulfil the two requirements, i.e., that $f_!$ reduces to the alternating sum of cohomology groups and is a homomorphism, the map $f_!$ is defined from $K(X)$ to $K(Y)$, where

$$K(X) = \frac{E(X)}{Q(X)}$$

($E(X)$ is the Free Abelian group generated by (coherent) sheaves over X. Elements of this group has the form $\sum_i z_i \mathcal{F}_i$ where z_i are integers and $\mathcal{F}_i$ are sheaves over X.)

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$Q(X)$ is generated by the expression $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$ whenever there is a short exact sequence of the form

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

$f_!$ is a homomorphism

For $\mathcal{F} = \mathcal{F}' + \mathcal{F}''$ in $K(X)$, $f_!$ should fulfil that

$$f_!(\mathcal{F}) = f_!(\mathcal{F}') + f_!(\mathcal{F}'')$$

Given the s.e.s $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, we obtain the l.e.s of the higher direct images. It then follows that

$$\sum_{i=0}^{q}(-1)^i R^i f_*(\mathcal{F}') - \sum_{i=0}^{q}(-1)^i R^i f_*(\mathcal{F}) + \sum_{i=0}^{q}(-1)^i R^i f_*(\mathcal{F}'') = 0$$

Interconnections and unifications

Previous talks have already mentioned some of Grothendieck's unifying concepts, e.g., his toposes and schemes.

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Historically, many other connections and unifications have been found and fruitfully exploited:

- Solving problems from plane geometry by algebraic tools.
- Finding tangents, extrema and areas of curves $\rightarrow$ Differential calculus.
- Graph-algebras connecting directed graphs and C^*-algebras.

Classifying interconnections — B. Mazur (2021)

- Ties (Bernoulli numbers connects Analysis, Number Theory, Homotopy Theory, Differential Topology, ...)
- Analogy (Prime numbers and Knots)
- Link (Langlands Program — duals)
- Bridge (Analytic Geometry, Geometric Algebra, Arithmetical Algebraic Geometry, ...)

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- **Links:** E.g., further properties of $K(X)$ exploit a bijective correspondence — link — to a similar construction for fiber spaces:

Borel & Serre (1958) establish that there is a bijective correspondence between $K(X)$ and $K_1(X)$.

This correspondence is used to prove various results, for example, that multiplication defined on $K(X)$ is associative (Borel and Serre 1958, p. 109).

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- **Link:** Further properties of $K(X)$ exploit a bijective correspondence — link — to a similar construction for fiber spaces.
- **Link:** The R-R theorems themselves establish formal correspondences between, e.g., cohomology groups and the Todd polynomial
- In addition are considerations of **internal coherence justifying** the introduction: Constructing the free abelian group and quotient, i.e. $K(X)$, to define the homomorphism $f_!$.

Concluding remarks

In addition to present one of Grothendieck's many impressive contributions, the talk concerns what can be said about the reality of mathematical entities.

We have seen that the postulation of mathematical objects is governed both by 'global' relations, or interconnections, as well as internal constraints.

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In addition to present one of Grothendieck's many impressive contributions, the talk concerns what can be said about the reality of mathematical entities.

We have seen that the postulation of mathematical objects is governed both by 'global' relations, or interconnections, as well as internal constraints.

We cannot create mathematical objects at will. We perceive analogies that motivate and when the right concepts have been introduced the results fall out — almost like magic.

But mathematics is not like fiction. A main difference is **the extent that relations play systematic roles in (the development of) mathematics.**

The end....

Thank you for your attention!