Investigating Definability in Propositional Logic via Grothendieck Topologies and Sheaves

Silvio Ghilardi
Università degli Studi di Milano

Chapman University, Orange (CA)

May 28, 2022

Aims of the Talk

- This talk brings together contributions made over the years concerning various topics in propositional logic; results have been obtained in cooperation with M. Zawadowski (the oldest ones) or L. Santocanale (the more recent ones).

Aims of the Talk

- This talk brings together contributions made over the years concerning various topics in propositional logic; results have been obtained in cooperation with M. Zawadowski (the oldest ones) or L. Santocanale (the more recent ones).
- We mostly consider definability question like: how could it be that a seemingly poor propositional language is in fact so rich and so expressive? As we will see, definability problems are also related to solving equations in appropriate free or extension algebras.

Aims of the Talk

- This talk brings together contributions made over the years concerning various topics in propositional logic; results have been obtained in cooperation with M. Zawadowski (the oldest ones) or L. Santocanale (the more recent ones).
- We mostly consider definability question like: how could it be that a seemingly poor propositional language is in fact so rich and so expressive? As we will see, definability problems are also related to solving equations in appropriate free or extension algebras.
- The above questions are formulated in syntactic terms; despite their purely symbolic nature, investigating them can take benefit from embeddings into geometric environments.

Aims of the Talk

- This talk brings together contributions made over the years concerning various topics in propositional logic; results have been obtained in cooperation with M. Zawadowski (the oldest ones) or L. Santocanale (the more recent ones).
- We mostly consider definability question like: how could it be that a seemingly poor propositional language is in fact so rich and so expressive? As we will see, definability problems are also related to solving equations in appropriate free or extension algebras.
- The above questions are formulated in syntactic terms; despite their purely symbolic nature, investigating them can take benefit from embeddings into geometric environments.
- Sheaves over Grothendieck topologies supply such environments, to be coupled with appropriate combinatorial components (Ehrenfeucht-Fraïssé Games).

Heyting Algebras

The quickest way to introduce Intuitionistic Propositional Logic (IPC) is via Heyting algebras

Heyting Algebras

The quickest way to introduce Intuitionistic Propositional Logic (IPC) is via Heyting algebras A Heyting algebra is a structure

$$\mathcal{H} = \langle H, \wedge, \vee, \bot, \top, \rightarrow \rangle$$

where $\langle H, \wedge, \vee, \bot, \top \rangle$ is a distributive lattice with zero and one and where the 'relative pseudocomplement' operation $\rightarrow$ satisfies the adjointness condition:

$$a \wedge b \leq c \iff b \leq a \rightarrow c$$

Heyting Algebras

The quickest way to introduce Intuitionistic Propositional Logic (IPC) is via Heyting algebras A Heyting algebra is a structure

$$\mathcal{H} = \langle H, \wedge, \vee, \bot, \top, \rightarrow \rangle$$

where $\langle H, \wedge, \vee, \bot, \top \rangle$ is a distributive lattice with zero and one and where the 'relative pseudocomplement' operation $\rightarrow$ satisfies the adjointness condition:

$$a \wedge b \leq c \Longleftrightarrow b \leq a \rightarrow c$$

Intuitionistic Propositional Logic is the set of formulae (built up from countably many variables using the connectives $\wedge, \vee, \bot, \top, \rightarrow$) which evaluate to $\top$ in any Heyting algebra, no matter how variables are interpreted as elements of the support of that algebras.

Heyting Algebras

In other words, we see formulae in (IPC) as terms in the equational theories of Heyting algebras. We use letetrs $t, u, \ldots$ for such formulae/terms.

Heyting Algebras

In other words, we see formulae in (IPC) as terms in the equational theories of Heyting algebras. We use letetrs $t, u, \ldots$ for such formulae/terms.

We write $t \vdash u$ to mean that the equation $t \to u = \top$ holds in any Heyting algebra (or, equivalently, in the free Heyting algebra over countably many generators).

Heyting Algebras

In other words, we see formulae in (IPC) as terms in the equational theories of Heyting algebras. We use letetrs $t, u, \ldots$ for such formulae/terms.

We write $t \vdash u$ to mean that the equation $t \to u = \top$ holds in any Heyting algebra (or, equivalently, in the free Heyting algebra over countably many generators).

The relation $t \vdash u$ is conveniently described by a suitable logical calculus (like natural deduction, sequent calculus, tableau calculus, etc.), but we do not need to care about the calculus (the problems we investigate are independent on a specified calculus).

Heyting Algebras

Heyting algebras are ubiquitous:

Heyting Algebras

Heyting algebras are ubiquitous:

- open sets of a topological space;

Heyting Algebras

Heyting algebras are ubiquitous:

- open sets of a topological space;
- Kripke frames (= downward closed subsets of a poset);

Heyting Algebras

Heyting algebras are ubiquitous:

- open sets of a topological space;
- Kripke frames (= downward closed subsets of a poset);
- subpresheaves of a presheaf;

Heyting Algebras

Heyting algebras are ubiquitous:

- open sets of a topological space;
- Kripke frames (= downward closed subsets of a poset);
- subpresheaves of a presheaf;
- subsheaves of a sheaf;

Heyting Algebras

Heyting algebras are ubiquitous:

- open sets of a topological space;
- Kripke frames (= downward closed subsets of a poset);
- subpresheaves of a presheaf;
- subsheaves of a sheaf;
- $\ldots$;

Heyting Algebras

Heyting algebras are ubiquitous:

- open sets of a topological space;
- Kripke frames (= downward closed subsets of a poset);
- subpresheaves of a presheaf;
- subsheaves of a sheaf;
- . . . ;

In all the above cases, the underlying lattice is complete and is a locale (infinite Joins distribute over finite meets); the relative pseudocomplement (as well as all other operations) is uniquely determined by the lattice order.

Heyting Algebras

The fact that the above mentioned Heyting algebras are complete, implies that extra structure is available on them.

Heyting Algebras

The fact that the above mentioned Heyting algebras are complete, implies that extra structure is available on them.

- Images and dual images along morphisms;

Heyting Algebras

The fact that the above mentioned Heyting algebras are complete, implies that extra structure is available on them.

- Images and dual images along morphisms;
- Least and Greatest Fixpoints (for monotonic endomaps);

Heyting Algebras

The fact that the above mentioned Heyting algebras are complete, implies that extra structure is available on them.

- Images and dual images along morphisms;
- Least and Greatest Fixpoints (for monotonic endomaps);
- Difference (dual of implication, in case the dual algebra is a locale).

Heyting Algebras

Suppose in fact that our Heyting algebra $\mathcal{H}_X$ is the Heyting algebras of sub-(pre)sheaves (of opens sets) of a (pre)sheaf (topological space) X and that we are given a natural transformation (open continuous map) $f : Y \longrightarrow X$, then we can compute images and dual images

$$\exists_f : \mathcal{H}_Y \longrightarrow \mathcal{H}_X \qquad \forall_f : \mathcal{H}_Y \longrightarrow \mathcal{H}_X$$

as left and right adjoints to the inverse image morphism $f^* : \mathcal{H}_X \longrightarrow \mathcal{H}_Y$.

Heyting Algebras

Suppose in fact that our Heyting algebra $\mathcal{H}_X$ is the Heyting algebras of sub-(pre)sheaves (of opens sets) of a (pre)sheaf (topological space) X and that we are given a natural transformation (open continuous map) $f : Y \longrightarrow X$, then we can compute images and dual images

$$\exists_f : \mathcal{H}_Y \longrightarrow \mathcal{H}_X \qquad \forall_f : \mathcal{H}_Y \longrightarrow \mathcal{H}_X$$

as left and right adjoints to the inverse image morphism $f^* : \mathcal{H}_X \longrightarrow \mathcal{H}_Y$. If $M : \mathcal{H}_X \longrightarrow \mathcal{H}_X$ is a monotonic map, we can compute the least fixpoint by (possibly transfinite) iterations

$$\bot \leq M(\bot) \leq M(M(\bot)) \leq \cdots$$

and similarly for the greatest fixpoint.

Heyting Algebras

Nevertheless, logicians are mostly interested in free algebras or maybe in finitely presented algebras, because they correspond to derivability in the pure calculus or in finitely axiomatixed theories.

Heyting Algebras

Nevertheless, logicians are mostly interested in free algebras or maybe in finitely presented algebras, because they correspond to derivability in the pure calculus or in finitely axiomatixed theories.

These are NOT complete.

Heyting Algebras

Nevertheless, logicians are mostly interested in free algebras or maybe in finitely presented algebras, because they correspond to derivability in the pure calculus or in finitely axiomatixed theories.

These are NOT complete.

Can we expect something similar to the above rich structure in such context?

Heyting Algebras

Nevertheless, logicians are mostly interested in free algebras or maybe in finitely presented algebras, because they correspond to derivability in the pure calculus or in finitely axiomatixed theories.

These are NOT complete.

Can we expect something similar to the above rich structure in such context?

The obvious answer should be NO, nevertheless....

Heyting Algebras

Nevertheless, logicians are mostly interested in free algebras or maybe in finitely presented algebras, because they correspond to derivability in the pure calculus or in finitely axiomatixed theories.

These are NOT complete.

Can we expect something similar to the above rich structure in such context?

The obvious answer should be NO, nevertheless....

THESE ARE OUR DEFINABILITY PROBLEMS.

Heyting Algebras

Nevertheless, logicians are mostly interested in free algebras or maybe in finitely presented algebras, because they correspond to derivability in the pure calculus or in finitely axiomatixed theories.

These are NOT complete.

Can we expect something similar to the above rich structure in such context?

The obvious answer should be NO, nevertheless....

THESE ARE OUR DEFINABILITY PROBLEMS.

In the final part of the talk we shall analyze the impact of the definability results on logical applications.

Heyting Algebras

We can generically formulate our problems as follows:

Heyting Algebras

We can generically formulate our problems as follows:

investigate the exactness properties of the (opposite of the) category of finitely presented Heyting algebras.

Heyting Algebras

We can generically formulate our problems as follows:

investigate the exactness properties of the (opposite of the) category of finitely presented Heyting algebras.

To do this, we need to embed our category $\mathcal{HA}_{fp}^{op}$ in a larger category (where images, fixpoints, etc. exist) and to find extra structure to recover our original category, via duality.

The Strategy

The dualities we need are specific for *finitely presented* algebras. These might be (at least partially) different from dualities for the category of all algebras.

The Strategy

The dualities we need are specific for *finitely presented* algebras. These might be (at least partially) different from dualities for the category of all algebras.

The dual of an algebra/theory is the space of its points/models (in the Boolean case, the dual of B is the set $Hom[B, 2]$ of the homomorphisms of B into the truth value algebra - this is nothing but the set of models of B, if we view the algebra B ‘as a theory’ - which is technically correct, modulo some explanations we omit).

The Strategy

The dualities we need are specific for *finitely presented* algebras. These might be (at least partially) different from dualities for the category of all algebras.

The dual of an algebra/theory is the space of its points/models (in the Boolean case, the dual of B is the set $Hom[B, 2]$ of the homomorphisms of B into the truth value algebra - this is nothing but the set of models of B, if we view the algebra B 'as a theory' - which is technically correct, modulo some explanations we omit).

In the Boolean case, if B is finitely presented, then B is finite and there is no need to put any further structure of the set $Hom[B, 2]$ to recover B.

The Strategy

The dualities we need are specific for *finitely presented* algebras. These might be (at least partially) different from dualities for the category of all algebras.

The dual of an algebra/theory is the space of its points/models (in the Boolean case, the dual of B is the set $Hom[B, 2]$ of the homomorphisms of B into the truth value algebra - this is nothing but the set of models of B, if we view the algebra B 'as a theory' - which is technically correct, modulo some explanations we omit).

In the Boolean case, if B is finitely presented, then B is finite and there is no need to put any further structure of the set $Hom[B, 2]$ to recover B.

However, going beyond the classical case, the situation becomes more involved: models must be structured!

The Strategy

The structure we have in mind has a geometric and a combinatorial component.

The Strategy

The structure we have in mind has a geometric and a combinatorial component.
The geometric structure is a sheaf structure; the combinatorial structure is the so-called bounded bisimulation and is defined via certain games.

The Strategy

The structure we have in mind has a geometric and a combinatorial component.
The geometric structure is a sheaf structure; the combinatorial structure is the so-called bounded bisimulation and is defined via certain games.
Our typical strategy goes as follows. Take the case of images:

The Strategy

The structure we have in mind has a geometric and a combinatorial component.
The geometric structure is a sheaf structure; the combinatorial structure is the so-called bounded bisimulation and is defined via certain games.
Our typical strategy goes as follows. Take the case of images:

- as models are structured as sheaves, if images exists, they must be sheaf-theoretic images;

The Strategy

The structure we have in mind has a geometric and a combinatorial component.

The geometric structure is a sheaf structure; the combinatorial structure is the so-called bounded bisimulation and is defined via certain games.

Our typical strategy goes as follows. Take the case of images:

- as models are structured as sheaves, if images exists, they must be sheaf-theoretic images;
- sheaf theoretic images are in fact 'definable' because they are closed under bounded (sufficiently high bounded!) bisimulation;

The Strategy

The structure we have in mind has a geometric and a combinatorial component.
The geometric structure is a sheaf structure; the combinatorial structure is the so-called bounded bisimulation and is defined via certain games.
Our typical strategy goes as follows. Take the case of images:

- as models are structured as sheaves, if images exists, they must be sheaf-theoretic images;
- sheaf theoretic images are in fact 'definable' because they are closed under bounded (sufficiently high bounded!) bisimulation;
- hence images exist in $\mathcal{HA}_{fp}^{op}$.

The Strategy

A similar strategy has been used for many other questions, for positive and negative results (definability of difference, existence of fixpoints via periodicity, regularity of epis and monos, characterization of projectivity, effectiveness of equivalence relations, etc.).

The Strategy

A similar strategy has been used for many other questions, for positive and negative results (definability of difference, existence of fixpoints via periodicity, regularity of epis and monos, characterization of projectivity, effectiveness of equivalence relations, etc.).

The geometric overview of the problems usually does not solve them (especially if they are non trivial), but indicates what one has to look for and how combinatorial arguments should finally be employed.

The geometric component

We present the duality for *finitely presented Heyting algebras* given in G.-Zawadowski book "Sheaf, games and model completions", Kluwer 2002.

The geometric component

We present the duality for *finitely presented Heyting algebras* given in G.-Zawadowski book "Sheaf, games and model completions", Kluwer 2002.

As geometric environment, we consider the category P_0 of finite rooted posets (with *p*-morphisms) and the category of sheaves over them with the canonical (Grothendieck) topology J_0.

The geometric component

We present the duality for *finitely presented Heyting algebras* given in G.-Zawadowski book "Sheaf, games and model completions", Kluwer 2002.
As geometric environment, we consider the category P_0 of finite rooted posets (with *p*-morphisms) and the category of sheaves over them with the canonical (Grothendieck) topology J_0.

A poset $(P, \leq)$ is rooted iff it has a greatest element ρ_P.

The geometric component

We present the duality for *finitely presented Heyting algebras* given in G.-Zawadowski book "Sheaf, games and model completions", Kluwer 2002.
As geometric environment, we consider the category P_0 of finite rooted posets (with p-morphisms) and the category of sheaves over them with the canonical (Grothendieck) topology J_0.

A poset $(P, \leq)$ is rooted iff it has a greatest element ρ_P.

$f : Q \longrightarrow P$ is a p-morphism iff it is order-preserving and moreover satisfies the following condition forall $q \in Q, p \in P$

$$p \leq f(q) \;\Rightarrow\; \exists q' \in Q \; (q' \leq q \;\&\; f(q') = p) \quad .$$

The geometric component

We present the duality for *finitely presented Heyting algebras* given in G.-Zawadowski book "Sheaf, games and model completions", Kluwer 2002.
As geometric environment, we consider the category P_0 of finite rooted posets (with p-morphisms) and the category of sheaves over them with the canonical (Grothendieck) topology J_0.

A poset $(P, \leq)$ is rooted iff it has a greatest element ρ_P.

$f : Q \longrightarrow P$ is a p-morphism iff it is order-preserving and moreover satisfies the following condition forall $q \in Q, p \in P$

$$p \leq f(q) \;\Rightarrow\; \exists q' \in Q \; (q' \leq q \;\&\; f(q') = p) \quad .$$

Covers are simple to describe here: C is a cover of P iff it contains a surjective map $f : Q \longrightarrow P$.

The geometric component

The typical sheaf we use is the sheaf of L-evaluations

$$h_L := \ Hom(-, L)$$

(the *Hom* is taken into the category of posets) for a finite poset $(L, \leq)$: in case L is the powerset of a finite set ordered by reverse inclusion, this is the sheaf of finite Kripke models (over a finite propositional language).

The geometric component

We have a functor

$$\Phi : \mathcal{HA}_{fp}^{op} \longrightarrow Sh(\mathsf{P}_0, \mathsf{J}_0)$$

sending a finitely presented Heyting algebra H to a sheaf

$$\Phi(H) \ = \ [\, P \mapsto \mathcal{HA}(H, \mathcal{D}(P))\,]$$

(i.e. $\Phi_\mathsf{H}(H)$ associates to every finite rooted poset P the set of all Heyting morphisms from H to the Heyting algebra $\mathcal{D}(P)$ of downward closed subsets of P). Both $\Phi(H)$ and Φ act on morphisms in the obvious way, by composition.

Φ is left exact and conservative.

The combinatorial component

In order to have a manageable description of the category of sheaves dual to finitely presented Heyting algebras, we introduce additional structure defined in terms of games.

The combinatorial component

In order to have a manageable description of the category of sheaves dual to finitely presented Heyting algebras, we introduce additional structure defined in terms of games.

It can be shown that, in the case of the sheaf of finite Kripke models, subsheaves correspond to sets of models closed under bisimulations.

The combinatorial component

In order to have a manageable description of the category of sheaves dual to finitely presented Heyting algebras, we introduce additional structure defined in terms of games.

It can be shown that, in the case of the sheaf of finite Kripke models, subsheaves correspond to sets of models closed under bisimulations.

It is well-known that there are sets of models closed under bisimulation that do not correspond to sets of models of any given formula.

The combinatorial component

In order to have a manageable description of the category of sheaves dual to finitely presented Heyting algebras, we introduce additional structure defined in terms of games.

It can be shown that, in the case of the sheaf of finite Kripke models, subsheaves correspond to sets of models closed under bisimulations.

It is well-known that there are sets of models closed under bisimulation that do not correspond to sets of models of any given formula.

Thus, for definability issues (i.e. for a full duality), subsheaves are too many, we need another ingredient, of a more combinatorial nature: bounded bisimulations.

The combinatorial component

In order to have a manageable description of the category of sheaves dual to finitely presented Heyting algebras, we introduce additional structure defined in terms of games.

It can be shown that, in the case of the sheaf of finite Kripke models, subsheaves correspond to sets of models closed under bisimulations.

It is well-known that there are sets of models closed under bisimulation that do not correspond to sets of models of any given formula.

Thus, for definability issues (i.e. for a full duality), subsheaves are too many, we need another ingredient, of a more combinatorial nature: bounded bisimulations.

Bounded bisimulations can be introduced either via a recursive definition or via Ehrenfeucht-Fraissé games.

Games and Bounded Bisimulations

Let $u : P \longrightarrow L$ and $v : Q \longrightarrow L$ be two L-evaluations.

Games and Bounded Bisimulations

Let $u : P \longrightarrow L$ and $v : Q \longrightarrow L$ be two L-evaluations.

The game we are interested in has two players, Player 1 and Player 2.

Games and Bounded Bisimulations

Let $u : P \longrightarrow L$ and $v : Q \longrightarrow L$ be two L-evaluations.

The game we are interested in has two players, Player 1 and Player 2.

Player 1 can choose either a point in P or a point in Q and Player 2 must answer by choosing a point in the other poset; the only rule of the game is that, if $\langle p \in P, q \in Q \rangle$ is the last move played so far, then in the successive move the two players can only choose points $\langle p', q' \rangle$ such that $p' \leq p$ and $q' \leq q$.

Games and Bounded Bisimulations

Let $u : P \longrightarrow L$ and $v : Q \longrightarrow L$ be two L-evaluations.

The game we are interested in has two players, Player 1 and Player 2.

Player 1 can choose either a point in P or a point in Q and Player 2 must answer by choosing a point in the other poset; the only rule of the game is that, if $\langle p \in P, q \in Q \rangle$ is the last move played so far, then in the successive move the two players can only choose points $\langle p', q' \rangle$ such that $p' \leq p$ and $q' \leq q$.

If $\langle p_1, q_1 \rangle, \ldots, \langle p_i, q_i \rangle, \ldots$ are the points chosen in the game, Player 2 wins iff for every $i = 1, 2, \ldots,$ we have that $u(p_i) = v(q_i)$.

Games and Bounded Bisimulations

We say that

- $u \sim_\infty v$ iff *Player 2 has a winning strategy* in the above game with infinitely many moves;
- $u \sim_n v$ (for $n > 0$) iff *Player 2 has a winning strategy* in the above game with n moves, i.e. he has a winning strategy provided we stipulate that the game terminates after n moves;
- $u \sim_0 v$ iff $u(\rho(P)) = v(\rho(Q))$ (recall that $\rho(P), \rho(Q)$ denote the roots of P, Q).

We shall use the notation $[v]_n$ for the equivalence class of an L-valuation v via the equivalence relation $\sim_n$.

The Duality Statement

We say that a subsheaf S of the evaluations sheaf h_L has b-index n iff it has the following property:

$$v \in S(P) \ \& \ v \sim_n u \quad \Rightarrow \quad v \in S(Q)$$

(P, Q are the domains of $v \in h_L(Q), u \in h_L(P)$). If $S \subseteq h_L$ has b-index n for some n, it is said to be definable.

The Duality Statement

We say that a subsheaf S of the evaluations sheaf h_L has b-index n iff it has the following property:

$$v \in S(P) \;\&\; v \sim_n u \quad \Rightarrow \quad v \in S(Q)$$

(P, Q are the domains of $v \in h_L(Q), u \in h_L(P)$). If $S \subseteq h_L$ has b-index n for some n, it is said to be definable.

Similarly a natural transformation among definable sheaves $S \subseteq h_L$ and $S' \subseteq h_{L'}$

$$\psi : S \longrightarrow S'$$

is said to *have b-index m* iff for every $v \in S(P)$ and $v' \in S(Q)$, we have that $v \sim_m v'$ implies $\psi_P(v) \sim_0 \psi_Q(v')$. Such a natural transformation is also said to be definable.

The Duality Statement

Theorem

The category of finitely presented Heyting algebras is dual to the subcategory of $Sh(\mathrm{P}_0, \mathrm{J}_0)$ *formed by definable sheaves and definable natural transformations.*

The Duality Statement

Theorem

The category of finitely presented Heyting algebras is dual to the subcategory of $Sh(\mathrm{P}_0, \mathrm{J}_0)$ *formed by definable sheaves and definable natural transformations.*

A definable sheaf is the sheaf of finite models of a propositional formula. The b-index is related to the nested implications in the formula.

The Duality Statement

Theorem

The category of finitely presented Heyting algebras is dual to the subcategory of $Sh(\mathsf{P}_0, \mathsf{J}_0)$ *formed by definable sheaves and definable natural transformations.*

A definable sheaf is the sheaf of finite models of a propositional formula. The b-index is related to the nested implications in the formula.

A definable natural transformation maps (via inverse image) definable sheaves to definable sheaves. Such a map is the dual of a substitution.

Image Closure

Theorem

Definable sheaves are closed under images and dual images along definable natural trasformations. Hence $\mathcal{HA}_{fp}^{op}$ is a Heyting category (and the functor Φ preserves the Heyting category structure).

Image Closure

Theorem

Definable sheaves are closed under images and dual images along definable natural trasformations. Hence $\mathcal{HA}_{fp}^{op}$ is a Heyting category (and the functor Φ preserves the Heyting category structure).

Theorem

Differences of subobjects exist in both $\mathcal{HA}_{fp}^{op}$ and $Sh(\mathsf{P}_0, \mathsf{J}_0)$ (and Φ preserves them). Thus, the opposite lattice of a finitely presented Heyting algebra is also a Heyting algebra.

Image Closure

The above theorems are proved via combinatorial facts about our games. For instance, closure under images requires the following Lemma:

Lemma

Let $f : C \longrightarrow D$ and let n be big enough to be a b-index for both f and C. Then there exists N such that whenever we have $v \sim_N f(u)$ for $u \in C_P$, $v \in D_Q$, there is $u' \in C_{P'}$, $u' \sim_n u$ such that $v \circ h = f(u')$, for some arrow $h : P' \longrightarrow Q$ in P_0.

The crucial ingredient in the proof is the notion of n-rank of an evaluation u: this is defined to be the cardinality non $\sim_n$-equivalent sub-evaluations obtained restricting u to the cone over a point $p \in dom(u)$.

Image Closure: Applications

We now investigates the logical meaning of the existence of images and dual images. This is equivalent to a Theorem by A. Pitts (1992):

Theorem

There is an interpretation of second order propositional intuitionistic calculus into ordinary intuitionistic calculus.

One can reformulate the above theorem also by saying that (IPC) enjoys uniform interpolation.

Image Closure: Applications

The interpretation of second order quantifiers maps an intuitionistic formula $\phi(x, \underline{y})$ to the intuitionistic formulae $\exists^x \phi(x, \underline{y}), \forall^x \phi(x, \underline{y})$ obtained as follows: i) one takes the definable sheaf corresponding to ϕ; ii) computes its image and dual images along suitable projections; iii) converts back such images and dual images into the formulae they define.

The above procedure is effective, because the number N of the above Lemma (which can be effectively computed as the double of a suitable maximum n-rank) gives also a search bound for implication nestings.

Image Closure: Applications

We also have a model theoretic reformulation of the images closure theorem:

Theorem

The first-order theory of Heyting algebras admits a model completion.

This model-theoretic reformulation can be better understood in terms of constraint solving (by constraint we mean a system of equations and inequations).

Image Closure: Applications

In fact, it turns out that the constraint

$$t_1(\vec{a}, x) = 1 \ \& \ \cdots \ \& \ t_n(\vec{a}, x) = 1 \ \& \ u_1(\vec{a}, x) \neq 1 \ \& \ \cdots \ \& \ u_m(\vec{a}, x) \neq 1$$

with parameters $\vec{a}$ from a Heyting algebra H is solvable in an extension of H iff the quantifier-free formula

$$(\exists^x \bigwedge_{i=1}^{n} t_i)(\vec{a}) = 1 \ \& \ (\forall^x (\bigwedge_{i=1}^{n} t_i \longrightarrow u_1))(\vec{a}) \neq 1 \ \& \ \cdots$$

$$\cdots \ \& \ (\forall^x (\bigwedge_{i=1}^{n} t_i \longrightarrow u_m))(\vec{a}) \neq 1$$

is true in H.

Existentially Closed Algebras

Let us call existentially closed an algebra H such that any constraint (with parameters from H) having a solution in a an extension of H has a solution in H itself.

Existentially Closed Algebras

Let us call existentially closed an algebra H such that any constraint (with parameters from H) having a solution in a an extension of H has a solution in H itself.

What we have seen is an infinite axiomatization for the theory of existentially closed Heyting algebras. The problem whether a finite axiomatization exists is still open.

Existentially Closed Algebras

Let us call existentially closed an algebra H such that any constraint (with parameters from H) having a solution in a an extension of H has a solution in H itself.

What we have seen is an infinite axiomatization for the theory of existentially closed Heyting algebras. The problem whether a finite axiomatization exists is still open.

[Darnière-Junker, Houston J. of Math., 2018] solved it positively for the 6 amalgamable locally finite varieties of Heyting algebras.

Existentially Closed Algebras

Let us call existentially closed an algebra H such that any constraint (with parameters from H) having a solution in a an extension of H has a solution in H itself.

What we have seen is an infinite axiomatization for the theory of existentially closed Heyting algebras. The problem whether a finite axiomatization exists is still open.

[Darnière-Junker, Houston J. of Math., 2018] solved it positively for the 6 amalgamable locally finite varieties of Heyting algebras.

[Carai-G., J. Symb. Log. 2019] solved it (also positively) for the case of Browverian Semilattices (i.e. the $\top, \wedge, \rightarrow$-fragment of intuitionistic logic).

μ-Calculus Over Intuitionistic Logic

We cannot directly apply our duality for investigating lowest and greatest fixpoints (because we are considering monotonic maps which are not Heyting algebras morphisms, such maps do not have duals in our setting).

μ-Calculus Over Intuitionistic Logic

We cannot directly apply our duality for investigating lowest and greatest fixpoints (because we are considering monotonic maps which are not Heyting algebras morphisms, such maps do not have duals in our setting).

This is what we want to investigate. The μ-calculus is obtained by adding to the language lowest amd greatest fixpoints

$$\mu x.\phi(x, \underline{y}), \qquad \nu x.\phi(x, \underline{y})$$

for (syntactically) monotonic (in x) formulae $\phi(x, \underline{y})$.

μ-Calculus Over Intuitionistic Logic

We cannot directly apply our duality for investigating lowest and greatest fixpoints (because we are considering monotonic maps which are not Heyting algebras morphisms, such maps do not have duals in our setting).

This is what we want to investigate. The μ-calculus is obtained by adding to the language lowest amd greatest fixpoints

$$\mu x.\phi(x, \underline{y}), \qquad \nu x.\phi(x, \underline{y})$$

for (syntactically) monotonic (in x) formulae $\phi(x, \underline{y})$.

Over (IPC), the μ-calculus collapses, as proved by [Mardaev, Algebra and Logic 1993], in the sense that $\mu x.\phi(x, \underline{y})$ and $\nu x.\phi(x, \underline{y})$ are always equivalent to plain intuitionistic formulae.

μ-Calculus Over Intuitionistic Logic

We cannot directly apply our duality for investigating lowest and greatest fixpoints (because we are considering monotonic maps which are not Heyting algebras morphisms, such maps do not have duals in our setting).

This is what we want to investigate. The μ-calculus is obtained by adding to the language lowest amd greatest fixpoints

$$\mu x.\phi(x, \underline{y}), \qquad \nu x.\phi(x, \underline{y})$$

for (syntactically) monotonic (in x) formulae $\phi(x, \underline{y})$.

Over (IPC), the μ-calculus collapses, as proved by [Mardaev, Algebra and Logic 1993], in the sense that $\mu x.\phi(x, \underline{y})$ and $\nu x.\phi(x, \underline{y})$ are always equivalent to plain intuitionistic formulae.

In the case of μx, this means that the sequence of formulae

$$\phi_0 := \bot, \quad \phi_1 := \phi(\phi_0/x, \underline{y}), \quad \phi_2 := \phi(\phi_1/x, \underline{y}), \cdots \tag{1}$$

becomes stationary (up to provable equivalence).

Ruitenburg Theorem

- We can deduce the collapse of μ-calculus from Ruitenburg Theorem: this is one of the most surprising results concerning intuitionistic propositional calculus (IPC).

Ruitenburg Theorem

- We can deduce the collapse of μ-calculus from Ruitenburg Theorem: this is one of the most surprising results concerning intuitionistic propositional calculus (IPC).
- It says the following:

Ruitenburg Theorem

- We can deduce the collapse of μ-calculus from Ruitenburg Theorem: this is one of the most surprising results concerning intuitionistic propositional calculus (IPC).
- It says the following:
- take a formula $\phi(x, \underline{y})$ of (IPC) (not necessarily one monotonic in x) and consider the sequence $\{\phi^i(x, \underline{y})\}_{i \geq 1}$ so defined:

$$\phi^1 :\equiv \phi, \quad \ldots, \quad \phi^{i+1} :\equiv \phi(\phi^i/x, \underline{y}) \tag{2}$$

Ruitenburg Theorem

- We can deduce the collapse of μ-calculus from Ruitenburg Theorem: this is one of the most surprising results concerning intuitionistic propositional calculus (IPC).
- It says the following:
- take a formula $\phi(x, \underline{y})$ of (IPC) (not necessarily one monotonic in x) and consider the sequence $\{\phi^i(x, \underline{y})\}_{i \geq 1}$ so defined:

$$\phi^1 :\equiv \phi, \quad \ldots, \quad \phi^{i+1} :\equiv \phi(\phi^i/x, \underline{y}) \tag{2}$$

- then, *taking equivalence classes under equivalence in (IPC), the sequence* $\{[\phi^i(x, \underline{y})]\}_{i \geq 1}$ *is ultimately periodic with period 2*.

Ruitenburg Theorem

- We can deduce the collapse of μ-calculus from Ruitenburg Theorem: this is one of the most surprising results concerning intuitionistic propositional calculus (IPC).
- It says the following:
- take a formula $\phi(x, \underline{y})$ of (IPC) (not necessarily one monotonic in x) and consider the sequence $\{\, \phi^i(x, \underline{y}) \,\}_{i \geq 1}$ so defined:

$$\phi^1 :\equiv \phi, \quad \ldots, \quad \phi^{i+1} :\equiv \phi(\phi^i/x, \underline{y}) \tag{2}$$

- then, *taking equivalence classes under equivalence in (IPC), the sequence* $\{\, [\phi^i(x, \underline{y})] \,\}_{i \geq 1}$ *is ultimately periodic with period 2*.
- The latter means that there is N such that

$$\vdash_{IPC} \phi^{N+2} \leftrightarrow \phi^N \quad . \tag{3}$$

Ruitenburg Theorem

- Since it is cleat that $\phi^i(\bot/x, \underline{y}) = \phi_i$ and since the sequence (1) is increasing, we have

$$\vdash \phi_N \to \phi_{N+1} \qquad \vdash \phi_{N+1} \to \phi_{N+2} \qquad \vdash \phi_N \leftrightarrow \phi_{N+2}$$

so that $\vdash \phi^N \leftrightarrow \phi^{N+1}$, proving the collapse.

Ruitenburg Theorem

- Since it is cleat that $\phi^i(\bot/x, \underline{y}) = \phi_i$ and since the sequence (1) is increasing, we have

$$\vdash \phi_N \to \phi_{N+1} \qquad \vdash \phi_{N+1} \to \phi_{N+2} \qquad \vdash \phi_N \leftrightarrow \phi_{N+2}$$

  so that $\vdash \phi^N \leftrightarrow \phi^{N+1}$, proving the collapse.
- Ruitenburg Theorem was shown in [Ruitenburg, J. Symb. Logic 1984] via a, rather involved, purely syntactic proof.

Ruitenburg Theorem

- Since it is cleat that $\phi^i(\bot/x, \underline{y}) = \phi_i$ and since the sequence (1) is increasing, we have

$$\vdash \phi_N \to \phi_{N+1} \qquad \vdash \phi_{N+1} \to \phi_{N+2} \qquad \vdash \phi_N \leftrightarrow \phi_{N+2}$$

  so that $\vdash \phi^N \leftrightarrow \phi^{N+1}$, proving the collapse.
- Ruitenburg Theorem was shown in [Ruitenburg, J. Symb. Logic 1984] via a, rather involved, purely syntactic proof.
- We supply a semantic proof [G.-Santocanale, Math. Str. Comp. Sci. 2020], using our duality and bounded bisimulations machinery.

Ruitenburg Theorem

- Since it is cleat that $\phi^i(\bot/x, \underline{y}) = \phi_i$ and since the sequence (1) is increasing, we have

$$\vdash \phi_N \to \phi_{N+1} \qquad \vdash \phi_{N+1} \to \phi_{N+2} \qquad \vdash \phi_N \leftrightarrow \phi_{N+2}$$

  so that $\vdash \phi^N \leftrightarrow \phi^{N+1}$, proving the collapse.
- Ruitenburg Theorem was shown in [Ruitenburg, J. Symb. Logic 1984] via a, rather involved, purely syntactic proof.
- We supply a semantic proof [G.-Santocanale, Math. Str. Comp. Sci. 2020], using our duality and bounded bisimulations machinery.
- Let us first analyze the (greatly simplified) case of classical logic.

The Algebraic Reformulation

In classical propositional calculus (CPC), Ruitenburg Theorem holds with index 1 and period 2, namely given a formula $\phi(x, \underline{y})$, we have that

$$\vdash_{CPC} \phi^3 \leftrightarrow \phi \tag{4}$$

The Algebraic Reformulation

In classical propositional calculus (CPC), Ruitenburg Theorem holds with index 1 and period 2, namely given a formula $\phi(x, \underline{y})$, we have that

$$\vdash_{CPC} \phi^3 \leftrightarrow \phi \tag{4}$$

The first step is to re-interpret this statement in the category of finitely presented Boolean algebras (actually, finitely generated free algebras would suffice).

The Algebraic Reformulation

Let us denote by $\mathcal{A}[x]$ the *algebra of polynomials* over $\mathcal{A}$, i.e. the coproduct of the Boolean algebra $\mathcal{A}$ with the free algebra on one generator (thus $\mathcal{F}_B(x, \underline{y})$ is equal to $\mathcal{F}_B(\underline{y})[x]$).

The Algebraic Reformulation

Let us denote by $\mathcal{A}[x]$ the *algebra of polynomials* over $\mathcal{A}$, i.e. the coproduct of the Boolean algebra $\mathcal{A}$ with the free algebra on one generator (thus $\mathcal{F}_B(x, \underline{y})$ is equal to $\mathcal{F}_B(\underline{y})[x]$).

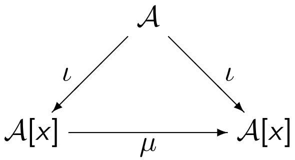
A slight generalization of statement (4) now reads as follows:

The Algebraic Reformulation

Let us denote by $\mathcal{A}[x]$ the *algebra of polynomials* over $\mathcal{A}$, i.e. the coproduct of the Boolean algebra $\mathcal{A}$ with the free algebra on one generator (thus $\mathcal{F}_B(x, \underline{y})$ is equal to $\mathcal{F}_B(\underline{y})[x]$).

A slight generalization of statement (4) now reads as follows:

- let $\mathcal{A}$ be a finitely presented Boolean algebra and let the map $\mu : \mathcal{A}[x] \longrightarrow \mathcal{A}[x]$ commute with the coproduct injection $\iota : \mathcal{A} \longrightarrow \mathcal{A}[x]$

  Then we have

$$\mu^3 = \mu \quad . \tag{5}$$

Dualization

The latter is a purely categorical statement, so that we can re-interpret it in dual categories.

Dualization

The latter is a purely categorical statement, so that we can re-interpret it in dual categories.

Finitely presented Boolean algebras are dual to finite sets; the duality functor maps coproducts into products and the free Boolean algebra on one generator to the two-elements set $2 = \{0, 1\}$.

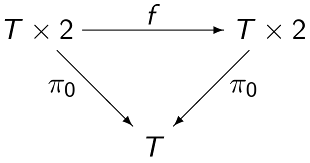
Dualization

The latter is a purely categorical statement, so that we can re-interpret it in dual categories.

Finitely presented Boolean algebras are dual to finite sets; the duality functor maps coproducts into products and the free Boolean algebra on one generator to the two-elements set $2 = \{0, 1\}$.

Thus statement (5) now becomes the following trivial exercise:

- Let T be a finite set and let the function $f : T \times 2 \longrightarrow T \times 2$ commute with the product projection $\pi_0 : T \times 2 \longrightarrow T$



Then we have

$$f^3 = f \quad . \tag{6}$$

Restating the Theorem for (IPC)

Considering that h_2 is the dual of the free algebra on one generator (2 is the 2-element chain), what we need to show is the following.

Restating the Theorem for (IPC)

Considering that h_2 is the dual of the free algebra on one generator (2 is the 2-element chain), what we need to show is the following.

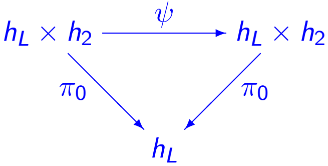
All natural transformations from $h_L \times h_2$ into itself, commuting over the first projection π_0 and having a b-index, are ultimately periodic with period 2.

Restating the Theorem for (IPC)

Considering that h_2 is the dual of the free algebra on one generator (2 is the 2-element chain), what we need to show is the following.

All natural transformations from $h_L \times h_2$ into itself, commuting over the first projection π_0 and having a b-index, are ultimately periodic with period 2.

Spelling this out, this means the following. Fix a natural transformation $\psi = \langle \pi_0, \chi \rangle : h_L \times h_2 \longrightarrow h_L \times h_2$ having a b-index such that the diagram



commutes; we have to find an N such that $\psi^{N+2} = \psi^N$.

A first approximation

It is useful, as a general strategy, to preminiarly study what happens keeping only the geometric structure (i.e. ignoring games and definability):

A first approximation

It is useful, as a general strategy, to preminiarly study what happens keeping only the geometric structure (i.e. ignoring games and definability):

Lemma

Let $\psi = \langle \pi_0, \chi \rangle : h_L \times h_2 \longrightarrow h_L \times h_2$ *be a natural transformation. Then for all rooted finite poset* P *there is* N_P *such that* $\psi^{N_P+2}(P) = \psi^{N_P}(P)$

A first approximation

It is useful, as a general strategy, to preminiarly study what happens keeping only the geometric structure (i.e. ignoring games and definability):

Lemma

Let $\psi = \langle \pi_0, \chi \rangle : h_L \times h_2 \longrightarrow h_L \times h_2$ *be a natural transformation. Then for all rooted finite poset* P *there is* N_P *such that* $\psi^{N_P+2}(P) = \psi^{N_P}(P)$

The proof is a moderate complication of what happens in the classical logic case (one can take N_P to be the height of P).

Ranks

Now the big jump:

Lemma

There is a (computable) N that does not depend on P in case ψ has a b-index.

Ranks

Now the big jump:

Lemma

There is a (computable) N that does not depend on P in case ψ has a b-index.

From this lemma, Ruitenburg's Theorem follows immediately. The lemma is proved via an appropriate notion of rank.

Some questions

The rank-based argument does not gove an optimal bound for N.

Some questions

The rank-based argument does not gove an optimal bound for N.

QUESTION: *is it possible to refine the refine semantic arguments and get an optimal bound for* N *?*

Some questions

The rank-based argument does not gove an optimal bound for N.

QUESTION: *is it possible to refine the refine semantic arguments and get an optimal bound for N ?*

QUESTION: In our paper we also show that there are free Heyting algebras endomorphisms which are not ultimately periodic. *Is it possible to characterize those which are such? and to give estimates for indexes and periods?*

Unification and Admissibility

Free algebras have special role in many logic applications. Solving a system of equations

$$(P) \qquad t_1 = u_1 \ \& \ \cdots \ \& \ t_n = u_n$$

in the countably generated free algebra means finding a substitution σ such that

$$\vdash t_1\sigma \leftrightarrow u_1\sigma \quad \& \quad \cdots \quad \& \quad \vdash t_n\sigma \leftrightarrow u_n\sigma$$

This is called the equational unification problem in computer science.

Unification and Admissibility

Proving that problem (P) has finitely many 'best solutions' (i.e. such that any other solution is an instance of such best ones) means showing that unification for Heyting algebras is finitary.

Unification and Admissibility

Proving that problem (P) has finitely many 'best solutions' (i.e. such that any other solution is an instance of such best ones) means showing that unification for Heyting algebras is finitary.

If unification is finitary, one can show that an inference rule

$$\frac{\gamma_1, \quad \ldots, \quad \gamma_n}{\delta} \ (R)$$

is admissible (i.e. does not alter the set of theorems) just by testing whether the finitely many best solutions of the conjunction of the antecedents produce theorems in (IPC) once applied to the conclusion.

Unification and Admissibility

Deciding admissibility of inference rules is an old problem by Friedman (1975).

Unification and Admissibility

Deciding admissibility of inference rules is an old problem by Friedman (1975).

The problem was first solved by Rybakov (1984).

Unification and Admissibility

Deciding admissibility of inference rules is an old problem by Friedman (1975).

The problem was first solved by Rybakov (1984).

The solution via finitarity of unification goes through a characterization of finitely presented projective Heyting algebras.

Unification and Admissibility

Deciding admissibility of inference rules is an old problem by Friedman (1975).

The problem was first solved by Rybakov (1984).

The solution via finitarity of unification goes through a characterization of finitely presented projective Heyting algebras.

This is another topic where our duality can help...

Characterizing duals of Projectives

Let C be a subsheaf of an evaluation sheaf h_L. We say that C has the extension property iff for every evaluation $v \in h_L(P)$ the following happens: if v_p (namely the restriction of v on the cone below p) belongs to C for all $p \in P$ different from the root of P, then there is $v' \in C$ such that $v'_p = v_p$ for all $p \in P$ different from the root of P.

Characterizing duals of Projectives

Let C be a subsheaf of an evaluation sheaf h_L. We say that C has the extension property iff for every evaluation $v \in h_L(P)$ the following happens: if v_p (namely the restriction of v on the cone below p) belongs to C for all $p \in P$ different from the root of P, then there is $v' \in C$ such that $v'_p = v_p$ for all $p \in P$ different from the root of P.

Theorem

A definable sheaf is dual to a finitely presented projective Heyting algebra iff it has the extension property. Such definable sheaves are closed under sheaf images.

Characterizing duals of Projectives

Let C be a subsheaf of an evaluation sheaf h_L. We say that C has the extension property iff for every evaluation $v \in h_L(P)$ the following happens: if v_p (namely the restriction of v on the cone below p) belongs to C for all $p \in P$ different from the root of P, then there is $v' \in C$ such that $v'_p = v_p$ for all $p \in P$ different from the root of P.

Theorem

A definable sheaf is dual to a finitely presented projective Heyting algebra iff it has the extension property. Such definable sheaves are closed under sheaf images.

It follows that every finitely generated Heyting algebra which is a a subalgebra of a finitely presented projective Heyting algebra is projective itself.

Back to Unification

A solution (or unifier) to the unification problem

$$(P) \qquad t_1(\underline{x}) = u_1(\underline{x}) \ \& \ \cdots \ \& \ t_n(\underline{x}) = u_n(\underline{x})$$

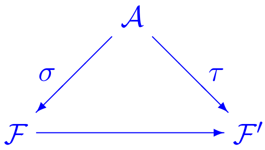
is (equivalently) a morphism

$$\sigma : \mathcal{A} \longrightarrow \mathcal{F}$$

among finitely presented algebras, where $\mathcal{F}$ is a free algebra and $\mathcal{A}$ is the free algebra over the $\underline{x}$ divided by the smallest congruence relation generated by the pairs in (P).

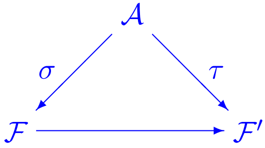
Back to Unification

A unifier σ is better than a unifier τ iff there is a commutative triangle

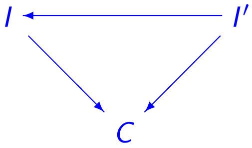


Back to Unification

A unifier σ is better than a unifier τ iff there is a commutative triangle



One can show that free algebras can be replaced by projective ones here. Hence we can dualize



where C, I, I' are the definable sheaves dual to $\mathcal{A}, \mathcal{F}, \mathcal{F}'$.

Back to Unification

We have seen that duals to projective are characterized by the extension property and that extension property is preserved by images.

Back to Unification

We have seen that duals to projective are characterized by the extension property and that extension property is preserved by images.

Thus we can replace I, I' by their sheaf images, because images "are better unifiers", according to the definition.

Back to Unification

We have seen that duals to projective are characterized by the extension property and that extension property is preserved by images.

Thus we can replace I, I' by their sheaf images, because images "are better unifiers", according to the definition.

In addition, comparison of such 'image unifiers' can be done via inclusions.

Back to Unification

We have seen that duals to projective are characterized by the extension property and that extension property is preserved by images.

Thus we can replace I, I' by their sheaf images, because images "are better unifiers", according to the definition.

In addition, comparison of such 'image unifiers' can be done via inclusions.

We only need a final fact: taking closure under $\sim_n$ of a subsheaf with extension property maintains the extension property.

Back to Unification

We have seen that duals to projective are characterized by the extension property and that extension property is preserved by images.

Thus we can replace I, I' by their sheaf images, because images "are better unifiers", according to the definition.

In addition, comparison of such 'image unifiers' can be done via inclusions.

We only need a final fact: taking closure under $\sim_n$ of a subsheaf with extension property maintains the extension property.

Thus "the best unifiers" have to be found among the definable subsheaves of C having the extension property and having a b-index less or equal to the b-index of C. Since there are only finitely many of them, this proves finitarity of unification and solves also Friedman problem on admissibility.

Conclusions

People working in propositional logics make extensive work to investigate properties having some combinatorial flavour or that can be proved by tools having some combinatorial flavour.

Conclusions

People working in propositional logics make extensive work to investigate properties having some combinatorial flavour or that can be proved by tools having some combinatorial flavour.

Such work often has important applications, but methods employed are usually rather ad hoc.

Conclusions

People working in propositional logics make extensive work to investigate properties having some combinatorial flavour or that can be proved by tools having some combinatorial flavour.

Such work often has important applications, but methods employed are usually rather ad hoc.

What I tried to show in my talk is that it is worth trying to perform that work in some conceptual framework.

Conclusions

People working in propositional logics make extensive work to investigate properties having some combinatorial flavour or that can be proved by tools having some combinatorial flavour.

Such work often has important applications, but methods employed are usually rather ad hoc.

What I tried to show in my talk is that it is worth trying to perform that work in some conceptual framework.

This is not meant to replace specific tools and techniques; rather it is meant to connect results and techniques to robust mathematical practice.

Conclusions

People working in propositional logics make extensive work to investigate properties having some combinatorial flavour or that can be proved by tools having some combinatorial flavour.

Such work often has important applications, but methods employed are usually rather ad hoc.

What I tried to show in my talk is that it is worth trying to perform that work in some conceptual framework.

This is not meant to replace specific tools and techniques; rather it is meant to connect results and techniques to robust mathematical practice.

To this aim, Grothendieck legacy might be quite precious.

THANKS FOR ATTENTION !