Grothendieck's homotopy hypothesis

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Definition

If X is a topological space, we define its foundamental groupoid $\Pi_1(X)$. The objects of $\Pi_1(X)$ are the points of X, and maps from x to y are homotopy classes of continuous path from x to y in X.

Theorem

The Π_1 construction induces an equivalence of categories between:

- *The category of groupoids with isomorphism class of functors between them.*
- *the category of 1-truncated spaces (CW-complexes) , that is spaces whose π_i groups are all trivial for $i > 1$, with homotopy classes of maps between them.*

A version of this for "2-groupoids" and 2-truncated spaces.

Theorem (Whitehead, 1949)

The category of 2-truncated spaces and homotopy class of maps between then is equivalent to the category of strict 2-groupoids and isomorphisms class of functor between them.

The *homotopy hypothesis* is the generalization of this as a correspondence between the category of n-truncated spaces and n-groupoids, and at the limit between arbitrary spaces and ∞-groupoids.

Grothendieck was very interested in this question in the early 80', and wrote about it in his manuscript "Pursuing Stacks" - and as far as I know he was one of the first to explicitely suggest it, or at leat to write about it.

To clarify, the general idea should be as follows: The fundamental ∞-groupoid $\Pi_\infty(X)$ of a space X should have:

- As objects the points of X,
- as arrow the continuous path in X,
- as 2-arrows the (end-point preserving) homotopy between paths,
- as three arrows the (boundary preserving) homotopy between homotopies,
- ..., more generally, as n-arrows the boundary preserving homotopies between $n-1$ arrows.

Of course this should be equipped with a lot of "composition operations" encoding the composition of paths, of homotopies, (in various directions).These composition making into an "∞-groupoids".
Note : Here the n-arrows are n-dimensional balls in X, whose source and target are the nothern and southern hemisphere.

One of the big problem to make this formal (and what motivated Grothendieck) is "what exactly is an ∞-groupoid ?"

This problem is generally thought of as a "test" for a possible definition of ∞-groupoids (hence the name "hypothesis" instead of conjecture or theorem).

Grothendieck doesn't mention it explicitely, but it seems reasonable to think he had in mind the developement of a theory of ∞-categories.

"Strict n-categories" are relatively easy to define:

Definition

A (strict) n-category is a category enriched in the category of (strict) $n-1$-categories.

That is, an n-category is a category, where for each pair of objects x, y the set of morphisms is promoted to be an $(n-1)$-category $Hom(x, y)$. The composition operation being an $(n-1)$-funtor between the $(n-1)$-categories $Hom(x, y) \times Hom(y, z) \to Hom(x, z)$.

And one can easily turn this inductive definition into a concrete algebraic definition with operations and axioms.

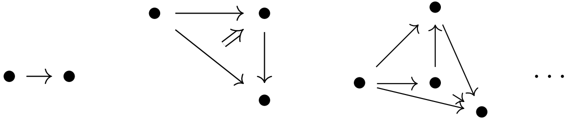
Strict ∞-categories can be easily defined by taking a limit in the definition of strict n-categories (very easy in the explicit algebraic definition) and strict ∞-groupoids are easily definable as those ∞-categories in which "every arrows in every dimension has an inverse".

Unfortunately, strict ∞-groupoids don't work for the homotopy hypothesis : when constructing $\Pi_\infty(X)$ the composition operation we can define on path and homotopies only satisfies the axioms of a strict ∞-category (or groupoid) up to homotopy, i.e. up to higher cells, and not up to equality.

Of course, one could hope that a different definition of $\Pi_\infty(X)$ would give a strict ∞-category - like I mentioned it is possible for 2-groupoid. But it can be shown this is impossible above dimension 3.

There are definitions of ∞-groupoids for which the homotopy hypothesis is well known : For exemple the most generally accepted point of view nowdays is that an ∞-groupoid is a Kan complex, that is a simplicial sets satisfying some filling conditions and that the Π_∞ functor is the simplicial nerve functor.

This corresponds to representing higher cells using simplicies instead of a globes:



And with this definitions, the homotopy hypothesis has been known since the work of Kan in the 50s (and Quillen in the 60s).

Grothendieck did not like the idea of taking Kan complexes as the definition of ∞-groupoids. He wanted a definition more in line with what we discussed earlier. So he proposed his own definition.

Definition

A Globular set is a collection of sets $X_0, X_1, \ldots\ldots$ with maps:

$$X_0 \overset{s_0,t_0}{\leftleftarrows} X_1 \overset{s_1,t_1}{\leftleftarrows} X_2 \overset{s_2,t_2}{\leftleftarrows} X_3 \overset{s_3,t_3}{\leftleftarrows} X_4 \cdots$$

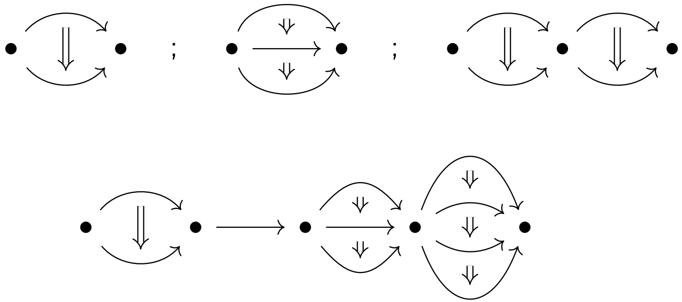
Such that $s_{i-1}s_i = s_{i-1}t_i$ and $t_{i-1}s_i = t_{i-1}t_i$

This encodes the idea of "Higher graphs" with objects, 1-arrows between objects, ..., n-arrows between pairs of parrallel $(n-1)$-arrows, but with no composition operations.

He then defines a category of “admissible diagrams” as a full subcategory of globular sets. Its objects represent the diagram we want to be able to compose in an ∞-category :

- In dimension 0: $D_0 = \bullet$
- In dimension 1: $D_1 = \bullet \to \bullet \quad ; \quad \cdots \quad ; \quad \bullet \to \cdots \to \bullet$
- In dimension 2:

$D_2 =$



We denote by C the category of these diagrames (a full subcategory of globular set). If X is a globular set and K is such a diagram, one can define $X(K)$ as the set of maps from K to X, that is the set of ways to evaluate the cells of K as cells of X in a consistant way.

This identifies globular sets with certain presheaf on C.

Grothendieck then construct a new category C_∞ with a bijective on object functor $C \to C_\infty$ and defines an ∞-groupoids to be a globular set, endowed with an extention to C_∞.

So the arrows in C_∞ correspond to "operations" that compose diagrames as above (especially the arrows whose domain is a globes) in an ∞-groupoid.

C_∞ is constructed inductively as the union of an increasing sequence:

$$C = C_0 \subset C_1 \subset C_2 \subset \cdots \subset \cdots \subset C_n \subset C_{n+1} \subset \cdots \subset C_\infty$$

Which are generated by an inductive principle : C_i is freely generated from C_{i-1} by the fact that:

- The functor $C \to C_i$ preserves the pushouts that corresponds to gluing diagram.
- for each pairs of arrows $f, g : D_n \rightrightarrows X$ in C_{i-1} whose restriction to D_{n-1} are equals, there is an arrow $h : D_{n+1} \to X$ in C_i such that $h \circ s = f$ and $h \circ t = g$.

(The construction does depends on some choices : for which pairs f, g do you freely add a h and for which you construct it from other ways)

To help make sense of this definition, let's consider some arrow in C_∞.

- In C_0, one consider the two maps from D_0 to $A_2 = (\bullet \to \bullet \to \bullet)$, sending the unique object of D_0 to either the leftmost or the rightmost object of A_2. Hence, in C_1 there is a map $(\bullet \to \bullet) \to (\bullet \to \bullet \to \bullet)$ preserving the end points. This encodes the composition of 1-cells.
- Using the map above, one can construct in C_1 two different maps from D_1 to $A_3 = (\bullet \to \bullet \to \bullet \to \bullet)$ corresponding to the two different way of bracketing the composition of three arrows. Hence in C_2, we obtain a map $D_2 \to A_3$ that corresponds to the operations which to each triplet of composable arrow associate an associativity 2-cells.

- In C_0 one can consider twice the identity maps $D_0 \to D_0$. This produces in C_1 a map $D_1 \to D_0$ that gives an operation sending each object x to an arrow $x \to x$. This is the "identity arrow".
- In C_0 one can consider the two maps $D_0 \to D_1$, but put in the wrong order. This produce in C_1 a map $D_1 \to D_1$ that turn an arrow $x \to y$ into an arrow $y \to x$. This corresponds to the inverse.
- One can construct in C_2 arrows with domain D_2 that corresponds to the 2-cells that witness that the composite of an arrow with its inverse is equivalent to the identity, and that the composite of an arrow with the identity is equivalent to the arrow.

Finally, Grothendieck constructs a "geometric realization" functor from C_∞ to the category of spaces that sends each object K to its obvious geometric realization $|K|$.

The image of arrows are defined by induction, using that each object goes to a contractible space as a key point of the construction.

One can then use this to define $\Pi_\infty(X)$ as the presheaf on C_∞ defined by $K \mapsto Hom(|K|, X)$.

Finally, one can define homotopy groups of ∞-groupoid relatively easily and show that the Π_∞-functor defined above preserve homotopy groups.

One defines weak equivalence of ∞-groupoids as maps that induces bijections on all homotopy groups, and the homotopy category of ∞-groupoid is defined by formally inverting this week equivalence. And finally Grothendieck conjecture:

Conjecture

Π_∞ *induces an equivalence of categories:*

$$Ho(Spaces) \to Ho(\infty\text{-}Groupoids)$$

Why do we care about this problem ?

Grothendieck ∞-groupoids are relatively hard to work with, so we do not expect this to be useful to study spaces. As we mentioned, other version of the homotopy hypothesis are already proved, and they have been extended to definition of higher categories, essentially circumventing Grothendieck's approach.

Essentially, this is a "test problem" for our understanding of higher structures in general. Essentially, this is one of the simplest exemple of a higher structure that "should work" but we are unable to work with. And there are several way to "weaken" the definition of strict ∞-categories, but several of them are not shown to be equivalent. The hope is that similar methods can be applied to other kinds of higher structures.

Where are we toward a proof of this conjecture ? - well this is still open, and it proved harder than what Grothendieck could have reasonably anticipated when he formulated it. But I think we are getting closer.

Grothendieck's conjecture appear at the begining of Pursuing stacks (1983), but this text wasn't easily available nor easy to read, and the precise nature of the conjecture stayed largely unknown for a long time.

In 1998 Batanin gave a definition of weak ∞-categories that was of the same flavor as Grothendieck's definition (globular set with composition operations). He also phrases his own version of the conjecture, which is very similar to Grothendieck's.

There are two papers proving partial results toward Batanin version of the conjecture. By Berger in 2001 and Cisinski in 2006. Unfortunately, there are mistakes in Berger paper that invalidates the results of both papers.

In 2007, Maltsiniotis published a preprint presenting Grothendieck homotopy hypothesis with all its details. In the same paper he also generalises Grothendieck definition of ∞-groupoid to a definition of ∞-categories.

In 2010, Ara (PhD thesis) proved that Batanin and Grothendieck-Maltsiniotis definition of ∞-categories are equivalent. This is an actual equivalence of ordinary categories between the algebraic structure, not a "up to homotopy" equivalences.

In 2013, Ara published a detailed study of the homotopy theory of Grothendieck ∞-groupoids. This includes many basic properties of homotopy groups, and a proof that weak equivalence satisfies 2-out-of-3. This is actually as far as I'm concerned the most technically difficult proof on the topic.

In his Phd thesis (published in 2018), Lanari studied the possibility of setting up a Quillen model category of ∞-groupoids. He gave a series of equivalent statement to the existence of this model structure. He also proved the model structure exists for 3-groupoids.

In 2016, (independently) I gave a proof of the homotopy hypothesis under the assumption of one of the condition of Lanari. Using the same methods I also proved the homotopy hypothesis for a different definition of ∞-groupoids which is also in the form "globular sets with composition operations".

In 2019, Lanari and myself proved that my 2016 conditional proof also works for n-groupoids (for finite n). In particular, we can prove the homotopy hypothesis for 3-groupoids, using Lanari's previous result.

Quillen model structure are our best tool to define well behaved homotopy theories. In fact, Grothendieck already raised the question of forming a model category of ∞-groupoids in pursuing stack.

We have a model structures for strict ∞-groupoids (The Brown-Golanski model structure) and strict ∞-categories (Lafont-Métayer-Worytkiewicz). Which give a candidate for a model structure on Grothendieck ∞-groupoids.

In order to show that this model structure there are two hard steps:

- Weak equivalence satisfies 2-out-of-3 (done by Ara).
- Pushouts of the maps $D_n \to D_{n+1}$ are weak equivalences.

This second point sounds like it should be very easy. But is still open at this point.

Lanari also observed that you can deduce this from the existence of an "∞-groupoids of arrows". That is given an ∞-groupoid X we want a construction of an ∞-groupoid "PX" whose objects are the arrows of X, morphisms are commutative square, and so one... Lanari gave a precise definition of PX as a globular set, but it is unknown how to make it an ∞-groupoid.

And as mentioned above, Lanari constructed these composition operation "by hand" for the case of 3-groupoids.

In my 2016 paper, I start from the following observation:

Ara showed (following Grothendieck) that if C is a model category (in which every object is fibrant) and Y is an object of C, then one can construct an adjunction $L : \infty\text{-Groupoid} \leftrightarrows C : R$ such that $L(D_0) = Y$.

I'm reading this as a sort of "universal property" of the category of ∞-groupoids. In order to make this precise there we need two things :

- Make the category of ∞-groupoids into a model category in which every object is fibrant, and such that $L \dashv R$ is a Quillen adjunction.
- Show some uniqueness property for this adjunction $L \dashv R$.

The first point, is essentially the model structure discussed on the previous slide. The second point can be dealt with : assuming the first point is proved, one can show that the adjunction $L \dashv R$ is unique up to homotopy equivalence.

This is enough to show the homotopy hypothesis by constructing other model categories with the same properties and using this universal property to show they are all equivalents.

Finally, in the same paper I constructed a model structure of "globular sets with operations" for which all this can be done. But in this version the "composition diagram" are no longer globular, but depends on previously constructed operations, for e.g. :

$$\begin{tikzcd}
 & \bullet \arrow[dd] \arrow[dr] & \\
\bullet \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Downarrow"] & {} \arrow[r, phantom, "\Downarrow"] & \bullet \\
 & \bullet \arrow[ur] &
\end{tikzcd}$$