AS IF CATEGORY THEORY WERE A FOUNDATION.

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AIMS

- CARVE OUT AN *AS-IF* INTERPRETATION OF MATHEMATICAL STRUCTURALISM BY DISENTANGLING *METHODOLOGICAL* CONSIDERATIONS FROM *METAPHYSICAL* ONES.
  - I BEGIN WITH PLATO TO SHOW THAT MUCH PHILOSOPHICAL MILK HAS BEEN SPILT OWING TO OUR *CONFLATING* THE METHOD OF MATHEMATICS WITH THE METHOD OF PHILOSOPHY.
  - I USE MY READING OF PLATO TO DEVELOP WHAT I CALL *AS-IFISM*, THE VIEW THAT, IN MATHEMATICS, WE TREAT OUR HYPOTHESES *AS IF* THEY WERE FIRST PRINCIPLES AND WE DO THIS WITH THE PURPOSE OF SOLVING MATHEMATICAL PROBLEMS *NOT* PHILOSOPHICAL ONES.
  - I EXTEND *AS-IFISM* TO MODERN MATHEMATICS WHEREIN THE METHOD OF MATHEMATICS BECOMES THE AXIOMATIC METHOD, NOTING THAT THIS ENGENDERS A SHIFT FROM AS IF HYPOTHESES TO AS IF AXIOMS.
  - I PAUSE TO NOTE THAT THE CONFLATION OF THE METHOD OF MATHEMATICS WITH THE METHOD OF PHILOSOPHY, WITNESSED WELL BY THE FREGE-HILBERT DEBATE, HAS LED TO THE CONTINUED *CONFUSION* OF MATHEMATICS WITH METAPHYSICS.
  - I PRESENT MY METHODOLOGICALLY INTERPRETED STRUCTURAL AS-IFISM
  - I SITUATE GROTHENDIECK WITHIN THIS ACCOUNT
  - I ARGUE THAT WE SHOULD TAKE CATEGORY THEORY *AS IF* IT WERE A FOUNDATION FOR MATHEMATICAL STRUCTURALISM

PLATO

- PLATO KEPT A CLEAR DISTINCTION BETWEEN MATHEMATICS AND METAPHYSICS AND THE KNIFE HE USED TO SLICE THE DIFFERENCE BETWEEN THE TWO WAS METHOD.

  - THE MATHEMATICIAN'S HYPOTHETICAL METHOD REASONS *DOWN FROM AN HYPOTHESIS* TOWARDS A CONCLUSION, WITH THE PURPOSE OF SOLVING A *MATHEMATICAL* OR *PHYSICAL* PROBLEM.

    - THE SOUL IS *FORCED TO* USE HYPOTHESES IN THE INVESTIGATIONS OF IT, NOT TRAVELING UP TO A FIRST PRINCIPLE, SINCE *IT CANNOT ESCAPE* OR GET ABOVE ITS HYPOTHESES, [AND SECOND] BY USING AS IMAGES [OR LIKENESSES] THOSE VERY THINGS OF WHICH IMAGES WERE MADE BY THE THINGS BELOW THEM, AND WHICH, BY COMPARISON TO THEIR IMAGES WERE THOUGHT TO BE CLEAR AND TO BE HONORED AS SUCH. (511A)

  - THE PHILOSOPHER'S DIALECTICAL METHOD REASONS *UP FROM AN HYPOTHESIS* TOWARDS A FIRST PRINCIPLE, WHICH TETHERS THE HYPOTHESIS, AND ONLY THEN DOES HE REASON DOWN TOWARDS A CONCLUSION, WITH THE PURPOSE OF SOLVING A *PHILOSOPHICAL* PROBLEM.

    - ALSO UNDERSTAND, THEN, THAT BY THE OTHER SUBSECTION OF THE INTELLIGIBLE I MEAN WHAT REASON ITSELF GRASPS BY THE POWER OF *DIALECTICAL DISCUSSION*, TREATING ITS HYPOTHESES, NOT AS FIRST PRINCIPLES [ABSOLUTE BEGINNINGS], BUT AS *GENUINE HYPOTHESES*, THAT IS, STEPPING STONES AND LINKS IN A CHAIN, IN ORDER TO ARRIVE AT WHAT IS *UNHYPOTHETICAL* AND THE *FIRST PRINCIPLE* OF EVERYTHING. HAVING GRASPED THIS PRINCIPLE, IT REVERSES ITSELF AND, KEEPING HOLD OF WHAT FOLLOWS FROM IT, COMES DOWN TO A CONCLUSION, MAKING NO USE OF ANYTHING VISIBLE AT ALL, BUT ONLY OF FORMS THEMSELVES, MOVING ON THROUGH FORMS TO FORMS, AND ENDING IN FORMS.(511B-C)

PLATO

- THE MATHEMATICIAN'S HYPOTHESES ARE THUS TAKEN *AS IF* THEY WERE TRUE FIRST PRINCIPLES, *BUT THEY ARE NOT.*
  - STUDENTS OF GEOMETRY, CALCULATION, AND THE LIKE *HYPOTHESIZE* THE ODD AND THE EVEN, THE VARIOUS FIGURES, THE THREE KINDS OF ANGLES, AND OTHER THINGS AKIN TO THESE IN EACH OF THEIR INVESTIGATIONS, *REGARDING THEM AS KNOWN*. THESE THEY TREAT THESE *AS IF* THEY WERE FIRST [ABSOLUTE] PRINCIPLES AND *DO NOT THINK IT NECESSARY TO GIVE ANY ARGUMENT FOR [ACCOUNT OF] THEM*, EITHER TO THEMSELVES OR TO OTHERS, *AS IF* THEY WERE EVIDENT TO EVERYONE. AND, CONSISTENTLY GOING FROM THESE FIRST PRINCIPLES THROUGH THE REMAINING STEPS, THEY ARRIVE [CONCLUDE] IN FULL AGREEMENT AT *THE POINT THEY SET OUT TO REACH IN THEIR INVESTIGATION*. (510C-D ITALICS ADDED.)

- THE PURPOSE OF THE MATHEMATICIANS' METHOD, WHICH *BEGINS WITH TAKING* HYPOTHESES *AS IF* THEY WERE FIRST PRINCIPLES, IS TO SOLVE MATHEMATICAL, METAMATHEMATICAL OR PHYSICAL PROBLEMS, *IT IS NOT* TO GIVE A PHILOSOPHICAL ACCOUNT OF THEM AS UNHYPOTHETICAL FIRST PRINCIPLES!

- AGAINST METAPHYSICAL REALISM: MATHEMATICS *DOES NOT NEED* A METAPHYSICS OF FORMS OR A DOMAIN OF STABLE MATHEMATICAL OBJECTS THAT, AS FIRST PRINCIPLES, ACCOUNTS FOR, OR TETHERS, THE TRUTH OF ITS HYPOTHESES.

THE CONFUSION

- THESE DIFFERENCES IN METHOD *DEMAND* DIFFERENCES IN *BOTH* EPISTEMOLOGY AND ONTOLOGY
  - THE MATHEMATICAL METHOD REQUIRES OBJECTS AS *OBJECTS OF THOUGHT* AND YIELDS *A KIND OF UNDERSTANDING*.
  - THE PHILOSOPHICAL METHOD REQUIRES OBJECTS AS OF *OBJECTS OF UNDERSTANDING* AND YIELDS *TRUE UNDERSTANDING* OR *KNOWLEDGE ITSELF*.

- EVEN GLAUCON IS SHOCKED TO HEAR THAT PHILOSOPHY AS A SCIENCE IS CLEARER THAN MATHEMATICS
  - I UNDERSTAND, THOUGH NOT ADEQUATELY – YOU SEE, IN MY OPINION, YOU ARE SPEAKING OF AN ENORMOUS TASK. YOU WANT TO DISTINGUISH THE PART OF WHAT IS AND WHAT IS INTELLIGIBLE, THE PART LOOKED AT BY THE SCIENCE OF DIALECTICAL DISCUSSION, AS *CLEARER THAN* THE PART LOOKED AT BY THE SO-CALLED SCIENCES – THOSE FOR WHICH HYPOTHESES ARE FIRST PRINCIPLES ... AND ALTHOUGH THOSE WHO LOOK AT THE LATTER PART ARE FORCED TO DO SO *BY MEANS OF THOUGHT* RATHER THAN SENSE PERCEPTION, STILL, BECAUSE THEY DO NOT GO BACK TO A GENUINE FIRST PRINCIPLE IN CONSIDERING IT, BUT PROCEED FROM HYPOTHESES, YOU DO NOT THINK THAT THEY HAVE TRUE UNDERSTANDING OF THEM, EVEN THOUGH ... THEY ARE INTELLIGIBLE. AND YOU SEEM TO ME TO CALL THE STATE OF MIND OF THE GEOMETERS – AND THE OTHERS OF THAT SORT – *THOUGHT* BUT *NOT UNDERSTANDING*; THOUGHT BEING INTERMEDIATE BETWEEN BELIEF AND UNDERSTANDING. (511C-D)

THE CONFUSION

- THE HYPOTHETICAL METHOD OF MATHEMATICS IS THUS *DISTINCT FROM* THE METAPHYSICAL METHOD OF PHILOSOPHY, AND, AS SUCH, SO IS ITS EPISTEMOLOGY, AND ITS ONTOLOGY.
    - THE MATHEMATICAL METHOD YIELDS A KIND OF UNDERSTANDING, THAT IS, YIELDS BELIEFS THAT ARE "RELIABLE GUIDES TO SOLVING PROBLEMS" (532B) BECAUSE THEY ARE BORN OUT OF *STABLE DEFINITIONS* AND A *RELIABLE METHOD*.
    - MATHEMATICAL OBJECTS ARE OBJECTS OF KNOWLEDGE BUT NOT OF UNDERSTANDING, THEY NOT AS REAL AS PHILOSOPHICAL OBJECTS, BUT, AS OBJECTS OF THOUGHT, THEY ARE STILL "CONCERNED WITH BEING" (534A).
    - ONLY THE METAPHYSICAL METHOD OF PHILOSOPHY YIELDS KNOWLEDGE ITSELF, THAT IS, YIELDS TRUE BELIEFS THAT ARE THEMSELVES FIXED TO, OR TETHERED BY, A DOMAIN OF *STABLE OBJECTS*, I.E., OBJECTS OF UNDERSTANDING OR FORMS.
- PHILOSOPHY *AS A SCIENCE* IS FOUNDED ON THE DIALECTICAL METHOD AND THE STABILITY OF ITS METAPHYSICAL OBJECTS, I.E., FORMS.
- MATHEMATICS *AS A SCIENCE* IS FOUNDED ON THE HYPOTHETICAL METHOD AND THE *STABILITY OF ITS DEFINITIONS*.

CORRECTING THE CONFUSION

- *THE CONFUSION*: AS PHILOSOPHERS OF MATHEMATICS, WE HAVE CONTINUED TO CONFLATE THE HYPOTHETICAL METHOD OF MATHEMATICS WITH THE METAPHYSICAL METHOD OF PHILOSOPHY.

- *THE CORRECTION*: WHEN I SAY A MATHEMATICAL OBJECT EXISTS, WHAT I MEAN IS THAT I TREAT MY HYPOTHESES *AS IF* THEY WERE TRUE FIRST PRINCIPLES AND, IN SO DOING, I ACT *AS IF* IT WERE TETHERED TO AN EXISTING OBJECT.
  - I RECOGNIZE, HOWEVER, THAT MY HYPOTHESES ARE *NOT* A FIRST PRINCIPLES AND SO MY OBJECT IS AN OBJECT OF THOUGHT (OF CONJECTURE) AND NOT OF KNOWLEDGE.

- BUT TOO I TREAT MY OBJECT *AS IF* IT EXISTS FOR THE PURPOSE OF SOLVING A MATHEMATICAL PROBLEM.
  - FOR EXAMPLE, IN SOLVING THE *MENO* PROBLEM, I TREAT THE LENGTH OF LINE THAT DOUBLES THE AREA OF A 2 UNIT SQUARE AS IF WERE AN OBJECT, BUT IT IS NOT. MOREOVER, IT IS ONLY BECAUSE OF THE STABILITY OF THE DEFINITIONS OF SQUARE AND OF DIAGONAL, TOGETHER WITH THE PYTHAGOREAN THEOREM, THAT I CAN REASON DOWN TO THE CONCLUSION THAT THE LENGTH WILL BE THE LENGTH OF THE DIAGONAL OF THE 2 UNIT SQUARE, *BUT* I CANNOT *KNOW* THE LENGTH OF THIS LINE AS A STABLE OBJECT SINCE IT IS $2\sqrt{2}$!!

PLATONIC METHODOLOGICAL AS-IFISM

- THUS, WE COME TO PLATO'S *METHODOLOGICAL AS-IF REALISM*:
  - IN MATHEMATICS, WE TREAT OUR HYPOTHESES *AS IF* THEY WERE TRUE FIRST PRINCIPLES, AND, *CONSEQUENTLY*, OUR OBJECTS *AS IF* THEY EXIST, AND WE DO THIS WITH THE PURPOSE OF SOLVING MATHEMATICAL, METAMATHEMATICAL AND PHYSICAL PROBLEMS.

- AS-IF MATHEMATICAL REALISM IS *DISTINCT FROM* METAPHYSICAL REALISM.
  - FOR THE AS-IF MATHEMATICAL REALIST, EXISTENCE IS A CONSEQUENCE OF TRUTH.
  - FOR THE METAPHYSICAL REALIST, TRUTH IS A CONSEQUENCE OF EXISTENCE.

MODERN AS-IFISM

- I NOW EXTEND THIS *METHODOLOGICAL AS-IFISM* TO MODERN MATHEMATICS WHEREIN THE METHOD OF MATHEMATICS BECOMES THE AXIOMATIC METHOD.

- THIS ENGENDERS A SHIFT FROM STARTING WITH *AS IF* HYPOTHESES TO STARTING WITH *AS IF AXIOMS*.

  - MATHEMATICS AS A SCIENCE IS FOUNDED ON THE AXIOMATIC METHOD AND THE STABILITY OF ITS DEFINITIONS, NOW IMPLICITLY EXPRESSED BY THE AXIOMS THEMSELVES.

- MATHEMATICS IS *NOT* FOUNDED ON THE STABILITY OF METAPHYSICAL OBJECTS.

THE FREGE-HILBERT DEBATE

- FREGE CONFUSES THE METHOD OF MATHEMATICS WITH THE METHOD OF PHILOSOPHY (WITH THE METHOD OF CONCEPT CONSTRUCTION), THAT IS, HE TAKES AXIOMS AS FIRST PRINCIPLES AND SO PRESUMES THAT WE NEED A STABLE DOMAIN OF OBJECTS TO TETHER OR "FIX" THEIR TRUTH.
  - FOR THE FREGEAN AXIOMS-AS-FIRST-PRINCIPLES ACCOUNT, THE PRIMITIVE TERMS EMPLOYED BY THE AXIOMS MUST BE DEFINED OVER A *FIXED* DOMAIN *BEFORE* THE STATEMENT OF THE AXIOMS. THAT IS, THEY MUST BE LOGICALLY CONSTRUCTED IN THE CASE OF ARITHMETIC AND KANTIAN CONSTRUCTED IN THE CASE OF GEOMETRY.

- HILBERT, HOWEVER, TAKES AXIOMS *AS IF* THEY WERE FIRST PRINCIPLES THAT THEMSELVES *IMPLICITLY DEFINE* OBJECTS, SO WHATEVER SATISFIES THE AXIOMS IS TAKEN AS AN OBJECT.
  - HILBERT TOOK AXIOMS AS *IMPLICIT DEFINITIONS* OVER A *VARIABLE* DOMAIN, SO THAT THE AXIOMS SYSTEMS THEMSELVES ARE BUT *SCHEMA* FOR DEFINING THOSE CONCEPTS EXPRESSED AS THE PRIMITIVE TERMS THAT ARE THEN VARIOUSLY INTERPRETED BY OBJECTS.

METAMATHEMATICAL IF-THENISM

- FOR FREGE THE STABILITY OF MATHEMATICAL DEFINITIONS WAS TO BE JUSTIFIED BY ASSUMING THE *TRUTH* OF THE AXIOMS, TRUTH AS FIXED LOGICALLY, IN THE CASE OF ARITHMETIC, OR TRUTH AS FIXED PHILOSOPHICALLY BY KANTIAN INTUITION, IN THE CASE OF GEOMETRY.
  - FREGE'S METAMATHEMATICAL ACCOUNT OF THE METHOD OF MATHEMATICS WAS: *IF* THE AXIOMS ARE TRUE, *THEN* THIS THEOREM CAN BE JUSTIFIED.

- FOR HILBERT, HOWEVER, THE STABILITY OF DEFINITIONS WAS JUSTIFIED BY ASSUMING THE *CONSISTENCY* OF THE AXIOMS. HENCE HILBERT'S FAMOUS QUOTE:
  - IF THE ARBITRARY POSTULATED AXIOMS DO NOT CONTRADICT EACH OTHER WITH THEIR COLLECTIVE CONSEQUENCES, THEN THEY ARE TRUE AND THE THINGS DEFINED BY MEANS OF THE AXIOMS EXIST. THAT, FOR ME, IS THE CRITERION OF TRUTH AND EXISTENCE.
  - HILBERT'S METAMATHEMATICAL ACCOUNT OF THE METHOD OF MATHEMATICS WAS: *IF* THE AXIOMS ARE CONSISTENT, *THEN* THIS THEOREM CAN BE JUSTIFIED.

IF-THENISM VERSUS AS-IFISM

- FEARING FORMALISM, BOTH FREGE AND HILBERT, CAME TO REJECT *LOGICAL IF-THENISM* AS A METAMATHEMATICAL ACCOUNT OF THE METHOD OF MATHEMATICS.

- WHAT I WILL NOW CONSIDER IS WHETHER THESE *LOGICALLY INTERPRETED IF-THENIST* VIEWS CAN BE WEAKENED TO THE *METHODOLOGICALLY INTERPRETED AS-IFIST* VIEW THAT PLATO SEEMED TO BE OFFERING UP.

LOGICAL IF-THENISM

- AS DETAILED BY RESNIK, FREGE DEVELOPED TWO FORMS OF IF-THENISM.

- ACCORDING TO THE FIRST *DEDUCTIVIST* OR *LOGICAL IF-THENIST* OPTION, MATHEMATICS IS IN THE BUSINESS OF ESTABLISHING RESULTS IN PURE LOGIC.

  - PRESUMING THAT
    - **A** STANDS FOR A QUANTIFICATIONAL SCHEMA DIAGRAMMING THE SUPPOSED AXIOMS AND **T** STANDS FOR A QUANTIFICATIONAL SCHEMA DIAGRAMMING THE SUPPOSED THEOREM OF THE THEORY. [RESNIK, 117]

  - THIS FIRST OPTION CAN EITHER BE EXPRESSED AS "**A** ⊃ **T**" IS LOGICALLY VALID (LOGICALLY PROVABLE) OR AS THE CLAIM THAT **T** IS A LOGICAL CONSEQUENCE OF **A** (**T** IS LOGICALLY DERIVABLE FROM **A**).

STRUCTURAL IF-THENISM

- ON THE SECOND *STRUCTURAL IF-THENIST* OPTION, HOWEVER, FREGE
  - VIEWS A MATHEMATICAL THEORY AS STUDYING THE PROPERTIES OF ALL STRUCTURES SATISFYING CERTAIN DEFINING CONDITIONS, BUT NEVER MAKES USE OF THE ASSUMPTION THAT SUCH STRUCTURES EXIST [RESNIK, 117]. THIS OPTION IS EXPRESSED AS "$A \vDash T$" (T IS LOGICALLY ENTAILED BY A).

- RESNIK NOTES THAT SUCH A STRUCTURALIST VIEW,
  - CAN RID MATHEMATICS OF ONTOLOGICAL PRESUPPOSITIONS WHILE RETAINING ITS APPARENT DESCRIPTIVE CHARACTER… [AND REDUCE] THE EPISTEMOLOGY OF MATHEMATICS TO THAT OF LOGIC, AND ACCOUNT FOR THE CENTRALITY OF PROOF IN MATHEMATICS [RESNIK, 118].

- NOTING YET ANOTHER VIRTUE, RESNIK FURTHER CLAIMS THAT THIS STRUCTURALIST OPTION OFFERS A STRAIGHTFORWARD ACCOUNT OF *APPLICABILITY*, THAT IS,
  - WHEN ONE FINDS A PHYSICAL STRUCTURE SATISFYING THE AXIOMS OF A MATHEMATICAL THEORY, THE APPLICATION OF THAT THEORY IS IMMEDIATE [RESNIK 118]

- RESNIK NEXT NOTES A FINAL VIRTUE, VIZ., THAT SUCH A STRUCTURAL IF-THENIST APPROACH IS IN-LINE WITH THE DEVELOPMENT OF ABSTRACT STRUCTURES, LIKE GROUP THEORY AND TOPOLOGY… AND CATEGORY THEORY!

PROBLEMS WITH STRUCTURAL IF-THENISM

- THE STRUCTURE PROBLEM (HELLMAN'S "HOME ADDRESS" PROBLEM)
  - WE NEED SET THEORY OR SOME OTHER THEORY AS A BACKGROUND THEORY OF STRUCTURES THEMSELVES.
    - HELLMAN'S *MODAL STRUCTURALIST* ACCOUNT OF *POSSIBLE* STRUCTURES.
    - SHAPIRO'S *PLATONIST ACCOUNT* OF *ACTUAL* STRUCTURES.

  - RESNIK IS QUICK TO REMIND US, HOWEVER, THAT THE STRUCTURAL IF-THENIST NEED *NOT*
    - MAKE ANY USE OF THE EXISTENCE ASSUMPTIONS OF SET THEORY … [SO ONE MAY] *REMAIN AGNOSTIC* WITH RESPECT TO THE EXISTENCE OF MATHEMATICAL STRUCTURES [RESNIK, 118].

  - THE BENEFITS OF SUCH AN AGNOSTIC *CARNAPIAN APPROACH* IS THAT IT
    - GIVES MATHEMATICS A *LINGUISTIC FRAMEWORK* WHICH IS REFERENTIAL… AND THUS AGREES WITH THE *PRIMA FACIE* REFERENTIAL CHARACTER OF MATHEMATICAL LANGUAGE AS USED BY PRACTICING MATHEMATICIANS [RESNIK, 118].

PROBLEMS WITH STRUCTURAL IF-THENISM

- **THE CONSISTENCY PROBLEM** (SHAPIRO'S "TURN TO LOGIC OR PHILOSOPHY" PROBLEM)
  - THE POSSIBLE INFINITE REGRESS OF RELATIVE CONSISTENCY PROOFS WILL ONLY BE STOPPED BY A TRUE THEORY. OTHERWISE, LIKE FREGE, WE NEED TO TURN TO LOGIC OR TO PHILOSOPHY TO ACCOUNT FOR TRUTH.

- FACED WITH THESE PROBLEMS, RESNIK PRESENTS US WITH TWO ALTERNATIVE PHILOSOPHICAL ROUTES:
  - 1. WE CAN TAKE THE *FREGEAN ROUTE* OF TURNING TO PHILOSOPHY AND BASE THE ASSUMPTION OF CONSISTENCY ON
    - A BELIEF IN THE MATHEMATICAL REALITY AND TRUTH OF SOME THEORY WHICH WILL VOUCH SAFE THE CONSISTENCY OF MATHEMATICAL THEORIES [RESNIK, 119].
  - 2. WE TAKE THE *CARNAPIAN ROUTE OF TURNING TO LOGIC* AND OFFER UP A RELATIVE CONSISTENCY PROOF TO
    - ARGUE THAT SINCE CONSISTENCY IS A MATHEMATICAL QUESTION, IT, TOO MUST BE TREATED DEDUCTIVELY ...[SO] THE ASSERTION THAT A GIVEN AXIOM SET IS CONSISTENT MUST ITSELF BE CONSTRUED AS CONDITIONAL UPON A BACKGROUND THEORY WITH RESPECT TO WHOSE TRUTH THE DEDUCTIVIST CAN REMAIN AGNOSTIC [RESNIK, 119].

METHODOLOGICAL STRUCTURAL AS-IFISM

- RESNIK, SHAPIRO, HELLMAN, AND METAPHYSICAL FOUNDATIONALISTS MORE GENERALLY, TAKE THE FREGEAN ROUTE, I OPT FOR THE CARNAPIAN, BUT INSTEAD OF TURNING TO LOGIC TO ANALYZE THE IF-THEN DEDUCTIVELY, I TURN TO MATHEMATICAL PROBLEMS TO ANALYZE THE AS-IF METHODOLOGICALLY.

- LET'S NOW PAUSE TO COMPARE *METAPHYSICALLY* INTERPRETED STRUCTURALISM WITH *METHODOLOGICALLY* INTERPRETED STRUCTURAL AS-IFISM AND SEE IF WE CAN'T FORESTALL THESE PROBLEMS.

  - LET'S RECALL THE BASIC PREMISE OF PLATO'S METHODOLOGICAL AS-IFISM: THAT MATHEMATICS IS USED TO SOLVE MATHEMATICAL, METAMATHEMATICAL AND PHYSICAL PROBLEMS AND THAT IT IS IN VIRTUE OF THESE USES THAT WE ARE JUSTIFIED IN TAKING OUR HYPOTHESES *AS IF THEY WERE TRUE*.

  - LIKEWISE, LET'S PRESUME THAT THE BASIC PREMISE OF METHODOLOGICAL STRUCTURAL AS-IFISM IS THAT MATHEMATICS IS USED TO SOLVE MATHEMATICAL, METAMATHEMATICAL AND PHYSICAL PROBLEMS AND THAT IT IS IN VIRTUE OF THESE USES THAT WE ARE MATHEMATICALLY JUSTIFIED IN TAKING A SET OF AXIOMS *AS IF* THEY WERE CONSISTENT FOR THE HILBERTIAN PURPOSE OF GIVING US TRUTH AND EXISTENCE, *AND* METAMATHEMATICALLY JUSTIFIED IN TAKING OUR BACKGROUND THEORY *AS IF* IT WERE TRUE FOR THE PURPOSE OF GIVING A RELATIVE CONSISTENCY PROOF.

METHODOLOGICAL STRUCTURAL AS-IFISM

- TAKING THIS METHODOLOGICAL AS-IFIST ROUTE, BY PLACING OUR FOCUS ON WHAT IS NEEDED FOR THE *PRACTICE* OF MATHEMATICS, WE ARE NEITHER COMMITTED TO THE UNCONDITIONAL CONSISTENCY OF OUR MATHEMATICAL AXIOMS NOR THE UNCONDITIONAL TRUTH OF OUR METAMATHEMATICAL BACKGROUND THEORY.

- BUT IS *METHODOLOGICAL* CONSIDERATIONS, AND *NOT* METAPHYSICAL ONES, THAT "CONDITION" OUR *AS IF* ASSUMPTIONS OF BOTH THE CONSISTENCY OF OUR MATHEMATICAL AXIOMS AND THE TRUTH OF OUR METAMATHEMATICAL BACKGROUND THEORY.

- THIS APPROACH THUS ALLOWS FOR A PLURALIST APPROACH TO MATHEMATICAL PRACTICE; ONE WHICH PUTS THE WORKING MATHEMATICIANS' METHODS, LOGICS AND THEORIES AT THE FOREFRONT OF ANY PHILOSOPHICAL ACCOUNT OF MATHEMATICAL PRACTICE.

- THERE IS NO ONE METHOD, ONE LOGIC, ONE THEORY OF MATHEMATICS; THERE IS NO ONE LOGICAL OR METAPHYSICAL ACCOUNT OF THE "WAY THINGS ARE" IN THE MATHEMATICAL REALM.

- THERE IS SIMPLY THE DESIRE TO SOLVE MATHEMATICAL, METAMATHEMATICAL AND PHYSICAL PROBLEMS
  - FOR EXAMPLE, WHILE THE CATEGORY AXIOMS CAN BE USED TO TALK ABOUT MATHEMATICAL STRUCTURES THEMSELVES, THIS IS ONLY IF THESE STRUCTURES ARE TAKEN AS "SMALL". OTHERWISE, AS GROTHENDIECK NOTED, WE WILL HAVE TO ADD YET ANOTHER METHODOLOGICAL METAMATHEMATICAL BACKGROUND THEORY TO SOLVE THIS PROBLEM, BE THIS A CLASS THEORY OR A THEORY OF UNIVERSES.

METHODOLOGICAL STRUCTURAL AS-IFISM

- WHAT WE GET, THEN, IS A METHODOLOGICAL VERSION OF STRUCTURAL AS-IFISM, MUCH LIKE MADDY'S *ENHANCED IF-THENISM*.

  - THAT IS, WE AGREE WITH MADDY [2019] THAT

    - MATHEMATICS IS A MATTER OF FIGURING OUT WHAT FOLLOWS FROM WHAT, WHERE THE CONCEPTS AND AXIOMS IN THE 'IF' PART ARE CHOSEN WITH AN EYE TO FACILITATING IMPORTANT MATHEMATICAL GOALS. [MADDY 21],

  - BUT WE DO NOT ANALYZE THE "WHAT FOLLOWS FROM WHAT" IN TERMS OF A LOGICAL DEDUCTIVIST IF-THEN READING, RATHER WE ANALYZE IT IN TERMS OF A METHODOLOGICAL STRUCTURALIST AS-IF READING.

METHODOLOGICAL STRUCTURAL AS-IFISM

- MY PROPOSED *METHODOLOGICAL AS-IFIST STRUCTURALIST* POSITION IS EVEN MORE ENHANCED THAN MADDY'S:
    - SOME OF OUR COMMITMENTS TO TAKING OUR AXIOMS AS IF THEY WERE FIRST PRINCIPLES, WILL BE MADE IN LIGHT OF *MATHEMATICAL PRACTICE*, WITH THE GOAL OF SOLVING *MATHEMATICAL PROBLEMS*,
    - SOME WILL BE MADE IN LIGHT OF *LOGICAL/PHILOSOPHICAL CONSIDERATIONS*, WITH THE GOAL OF SOLVING *METAMATHEMATICAL PROBLEMS*, AND
    - SOME WILL BE MADE IN LIGHT OF *MATHEMATICAL APPLICABILITY*, WITH THE GOAL OF SOLVING *PHYSICAL PROBLEMS*.

- *NONE* OF THESE COMMITMENTS, HOWEVER, WILL BE MADE WITH THE GOAL OF SOLVING *METAPHYSICAL PROBLEMS*, I.E., PROBLEMS ABOUT WHAT "FIXES" THE CONSISTENCY OF OUR AXIOMS OR "FIXES" THE TRUTH OF OUR BACKGROUND THEORY.

- AGAINST METAPHYSICAL STRUCTURALISM: MATHEMATICS *DOES NOT NEED* A METAPHYSICS OF STRUCTURES THAT "FIXES", OR CONDITIONS, THE TRUTH OF ITS AXIOMS.

GROTHENDIECK AND MATHEMATICAL PRACTICE

- LET'S PAUSE NOW TO SITUATE GROTHENDIECK AND HIS ACCOUNT OF *MATHEMATICAL PRACTICE* AS SEEN NOW FROM WITHIN MY AS-IFIST METHODOLOGICAL ACCOUNT

    - RECALL THAT ON MY VIEW MATHEMATICS AS A SCIENCE IS FOUNDED ON THE HYPOTHETICAL METHOD AND THE STABILITY OF ITS DEFINITIONS; IT IS *NOT* FOUNDED ON THE DIALECTIC METHOD AND THE STABILITY OF ANY METAPHYSICAL OBJECTS.

- THIS EXPLAINS GROTHENDIECK'S ATTENTION TO DEFINITIONS, I.E., TO HIS AIMS OF MAKING THEM MORE STABLE, BUT ALSO TO HIS VIEW THAT DEFINITIONS THEMSELVES EXPRESS BUT POINTS OF VIEW.

- BY ASKING NEW QUESTIONS, BY OFFERING UP NEW STABLE DEFINITIONS, BY ATTEMPTING TO SOLVE NEW PROBLEMS, WE COME TO SEE THAT THESE CAN CONVERGE TO GREAT IDEAS OR SINGLE THEMES THAT CAN BE USED TO SOLVE MORE PROBLEMS, AND THESE TO VISIONS OR SCHEMATIC THEMES (SCHEMES, TOPOS, SHEAVES) THAT CAN BE USED FOR FOUNDATIONAL WORK, I.E., TO SOLVE PROBLEMS BY UNIFYING OR ORGANIZING.

GROTHENDIECK AND MATHEMATICAL PRACTICE

- THE MATHEMATICIANS' MATHEMATICAL TASK, IS THUS FOURFOLD:

    - 1. TO SEE THE QUESTIONS THAT UNDERLIE MATHEMATICAL PROBLEMS,

    - 2. TO SEE THE NEEDED GOOD NOTIONS AND STATEMENTS THAT WILL MAKE THESE QUESTIONS PRECISE,

    - 3. TO GIVE EXPRESSION TO THESE AS IDEAS, VISIONS AND THEMES VIA STABLE DEFINITIONS THAT CLEARS THEM FROM THE VEILS OF MIST AND, FINALLY,

    - 4. TO USE THESE TO SOLVE PROBLEMS.

- THE MATHEMATICIAN'S PHILOSOPHICAL TASK IS TO DENY THAT THERE IS A MATHEMATICAL UNIVERSE, FIXED ONCE FOR ALL, BUT TO, NONETHELESS, ACT *AS IF* SHE WAS GRASPING MATHEMATICAL REALITY, DISCOVERING NOT CREATING; TO BORROW PLATO'S PHRASE, IT IS THIS ACT THAT WILL MAKE THE MATHEMATICIAN AN ENERGETIC SEEKER AFTER UNDERSTANDING.

GROTHENDIECK AND MATHEMATICAL PRACTICE

- THE ERROR OF BOURBAKI'S NOTION OF STRUCTURE, THEN, IS THAT IT IS A PURELY LINGUISTIC ORDERING OF IDEAS WITHOUT BRINGING ANY NEW IDEA OF ITS OWN, THIS NEW IDEA SHOULD THAT OF A RIGOROUS MATHEMATICAL DEFINITION OF THE NOTION OF "STRUCTURE"… BUT THIS IDEA IS AKIN TO THAT OF AN INTELLIGENT AND IMAGINATIVE LEXICOGRAPHER, THAN TO AN ELEMENT OF RENEWAL OF A LANGUAGE GIVING A NEW APPREHENSION OF REALITY.

- THE PREFERRED IDEA OF STRUCTURE, IS NOT PURELY LINGUISTIC, BUT RATHER IS AN EXPRESSED IDEA WHICH FURTHER WE TAKE *AS IF* IT WERE AN APPREHENSION OF A FIXED MATHEMATICAL REALITY, THIS TASK, WE ARE TOLD, WAS OVERTAKEN BY EVENTS, WITH THE INFLUX OF "CATEGORICAL" METHODS.

- SO IT IS BY LOOKING TO CATEGORICAL METHODS, THAT IS, BY TAKING OUR CATEGORICAL AXIOMS *AS IF* THEY WERE FIRST PRINCIPLES, THAT WE ARE TO FIND THE IDEA OF STRUCTURE EXPRESSED *AS IF* IT WERE A MATHEMATICAL OBJECT.

THE MATHEMATICAL STRUCTURALIST'S PROBLEMS

- WE MUST NOW TAKE UP THOSE *PHILOSOPHICAL CONSIDERATIONS* OF MATHEMATICAL STRUCTURALISTS, WHO TAKE MATHEMATICAL OBJECTS AS *NOTHING BUT POSITIONS IN A STRUCTURE*.
    - AS *AS-IF* MATHEMATICAL STRUCTURALISTS, WE ARE COMMITTED TO THESE OBJECTS *AS IF THEY EXIST*, BECAUSE WE ARE COMMITTED TO TAKING OUR AXIOMS *AS IF* THEY WERE TRUE FIRST PRINCIPLES.
    - AS *METHODOLOGICAL AS-IF* STRUCTURALISTS, WE ARE COMMITTED TO TAKING OUR AXIOMS *AS IF THEY WERE TRUE* FIRST PRINCIPLES IN VIRTUE OF THEIR USE IN SOLVING MATHEMATICAL, METAMATHEMATICAL AND PHYSICAL PROBLEMS.
- WE MUST NOW CONSIDER THE MATHEMATICAL PROBLEM OF HOW TO TALK ABOUT MATHEMATICAL *OBJECTS THEMSELVES*.
- FOLLOWING HILBERT'S MOVE FROM CONSISTENCY TO TRUTH TO EXISTENCE, AT THE MATHEMATICAL LEVEL, AN *OBJECT IS WHATEVER SATISFIES THE AXIOMS*, E.G., A NUMBER IS WHATEVER SATISFIES THE PA AXIOMS, A GROUP IS WHATEVER SATISFIES THE GROUP AXIOMS, A SET IS WHAT EVER SATISFIES THE SET AXIOMS, ETC.,

THE MATHEMATICAL STRUCTURALIST'S PROBLEMS

- NEXT, WE MUST CONSIDER THE METAMATHEMATICAL PROBLEM OF HOW TO TALK ABOUT MATHEMATICAL *STRUCTURES THEMSELVES*?

- THE METHODOLOGICAL MATHEMATICAL STRUCTURALIST'S METAMATHEMATICAL PROBLEM, THEN, IS HOW TO TALK ABOUT MATHEMATICAL STRUCTURES THEMSELVES, *WITHOUT* TURNING TO METHODS OF PHILOSOPHY, I.E., *WITHOUT* ASSUMING THAT SOME FOUNDATIONAL THEORY MUST BE TAKEN AS TRUE IN THE METAPHYSICAL SENSE OF THE TERM, I.E., AS UNCONDITIONALLY TRUE IN VIRTUE OF REFERENCE TO ACTUALLY OR POSSIBLY EXISTING STRUCTURES, CATEGORIES, SETS, MODELS, ETC.

- *MY SOLUTION*: ADOPT A METHODOLOGICAL *AS IF* STRUCTURALIST APPROACH TO THIS METAMATHEMATICAL PROBLEM AND TAKE CATEGORY THEORY *AS IF* IT WERE A FOUNDATION!!!

AS IF CATEGORY THEORY WERE A FOUNDATION

- AT THE *MATHEMATICAL* LEVEL, I.E., THE LEVEL THAT WE TALK ABOUT MATHEMATICAL STRUCTURES, STRUCTURES LIKE SET STRUCTURE, GROUP STRUCTURE, TOPOLOGICAL STRUCTURE, ETC.,

- WE TAKE THE THEORY DEFINED BY THE EM AXIOMS *AS IF* IT WERE A FOUNDATION
    - CATEGORY THEORY IS HERE TO BE TAKEN AS A CARNAPIAN LINGUISTIC FRAMEWORK/HILBERTIAN SCHEMA USED TO ORGANIZE WHAT WE SAY ABOUT MATHEMATICAL STRUCTURES AS CAT-STRUCTURED, E.G., STRUCTURES LIKE **SET, GRP, TOP,** ETC,.
    - THAT IS, THE EM AXIOMS, TAKEN *AS IF* THEY WERE CONSISTENT, ARE USED TO TALK ABOUT SETS, GROUPS, TOPOLOGICAL SPACES THEMSELVES TAKEN *AS IF* THEY WERE CAT-STRUCTURED.

AS IF CATEGORY THEORY WERE A FOUNDATION

- AT THE *SET-THEORETIC* LEVEL I.E., THE LEVEL THAT WE TALK ABOUT THE INTERNAL OR LOCAL SET-STRUCTURE OF MATHEMATICAL STRUCTURES.

- WE TAKE THE THEORY DEFINED BY THE ETCS AXIOMS AS IF IT WERE A FOUNDATION
  - CATEGORY THEORY IS HERE TO BE TAKEN AS A CARNAPIAN LINGUISTIC FRAMEWORK/HILBERTIAN SCHEMA USED TO ORGANIZE WHAT WE SAY ABOUT MATHEMATICAL STRUCTURES THEMSELVES AS INTERNALLY OR LOCALLY SET-STRUCTURED
  - THAT IS, THE ETCS AXIOMS, TAKEN AS IF THEY WERE CONSISTENT, ARE USED TO TALK ABOUT A TOPOS, A LOGIC, A QUANTIFIER, ETC., TAKEN AS IF THEY WERE SET-STRUCTURED.

AS IF CATEGORY THEORY WERE A FOUNDATION

- AT THE *METAMATHEMATICAL* LEVEL, I.E., THE LEVEL THAT WE TALK ABOUT *CATEGORIES THEMSELVES*.

- WE TAKE THE THEORY DEFINED BY THE CCAF AXIOMS *AS IF* IT WERE A FOUNDATION.
  - CATEGORY THEORY IS HERE TO BE TAKEN AS A CARNAPIAN LINGUISTIC FRAMEWORK/HILBERTIAN SCHEMA USED TO ORGANIZE WHAT WE SAY ABOUT CATEGORIES THEMSELVES AS CAT-STRUCTURED.
  - THAT IS, THE CCAF AXIOMS, TAKEN *AS IF* THEY ARE TRUE FOR THIS METAMATHEMATICAL PURPOSE, ARE USED TO TAKE CATEGORIES THEMSELVES *AS IF* THEY EXIST.
  - FOLLOWING OUR STRUCTURALIST SCHEMA, A CATEGORY IS WHATEVER SATISFIES THE CCAF AXIOMS.

AS IF CATEGORY THEORY WERE A FOUNDATION

- THUS, I COME TO THE STRUCTURALIST CONCLUSION THAT I HAVE ARGUED FOR IN MANY PAPERS BUT THAT I NOW PRESENT WITH A METHODOLOGICAL AS-IFIST TWIST:
  - TAKING CT *AS IF* IT WERE A FOUNDATION FOR MATHEMATICAL STRUCTURALISM SHOWS THAT IT IS POSSIBLE TO SPEAK *AS IF* STRUCTURE WERE PRIOR TO OBJECTS AND *AS IF* OBJECTS WERE NOTHING BUT POSITIONS IN A STRUCTURE, WITHOUT OUR HAVING TO THAT PRESUME CT, OR INDEED ANY THEORY, IS A METAPHYSICAL FOUNDATION.

  - AGAINST METAPHYSICAL STRUCTURALISM: A STRUCTURAL ACCOUNT OF MATHEMATICS DOES NOT NEED A FOUNDATION AS METAPHYSICS OF POSSIBLE OR ACTUAL STRUCTURES THAT, AS FIRST PRINCIPLES ACCOUNTS FOR, OR "FIXES", THE TRUTH OF ITS AXIOMS
  - ALL IT NEEDS IS THE LANGUAGE OF CATEGORY THEORY TAKEN *AS IF* IT WERE A FOUNDATION.