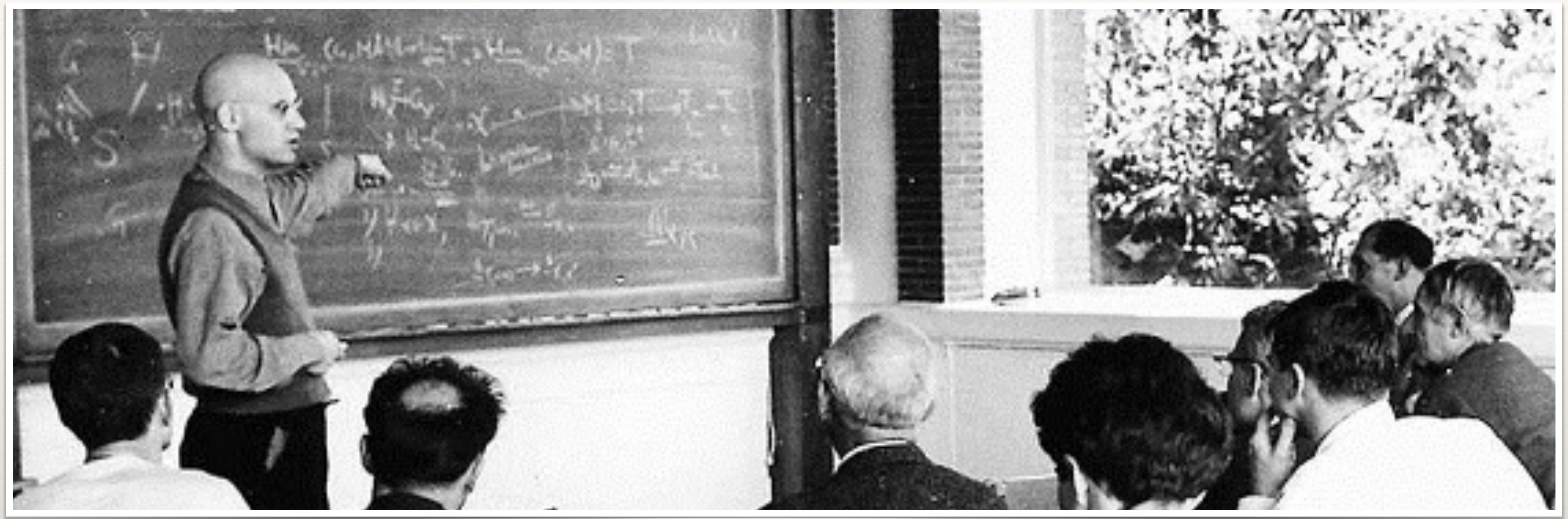


Grothendieck, a Multifarious Giant: Mathematics, Logic and Philosophy; Chapman University; May 24-28 2022

Grothendieck, Bourbaki and Mathematical Structuralism

Jean-Pierre Marquis
Département de philosophie
Université de Montréal
Qc, Canada

Warning

- This is part of a larger project on the rise and significance of mathematical structuralism in 20th century mathematics.
- Main goal: compare and contrast Bourbaki's mathematical structuralism with Grothendieck and, more generally, categorical mathematical structuralism from the logical, epistemological, semantical and ontological standpoints.
- This talk gives a bird's eyes view on important issues…

Bourbaki: a very, very short overview

- Bourbaki was born in 1934 -1935 with a simple goal in mind: write a modern treatise on analysis (in french).
- Start from scratch and build up from primitive notions.
- Thus the need to start with logic and sets, general topology, algebra, etc.
- It quickly becomes a singular enterprise, whose nature is difficult to capture.

Bourbaki's mathematical structuralism in a nutshell

- Bourbaki endorses from the very beginning a form of structuralism.
- It is described in *metamathematical* terms:
  - The notion of species of structures;
  - A *metalogical* requirement on *any* mathematical theories: if $P(X)$ and $X \simeq Y$, then $P(Y)$. (In Bourbaki's jargon: relations have to be *transportable*.)

- « Every *proposition* of the theory of the structures $\sigma \in \mathcal{T}$ will be transportable in the same way, by using the proper extensions, and will yield a *proposition* of the theory of structures belonging to $f(\mathcal{T})$. »

Bourbaki, 1939.

« Suffice it to say that the axiomatic studies of the nineteenth and twentieth centuries have gradually replaced the initial pluralism of the mental representations of these « beings » — … — by a unitary concept, gradually reducing all the mathematical notions, first to the concept of natural number and then, in a second stage, to the notion of set. (…) From this new point of view, mathematical structures become, properly speaking, the only « objects » of mathematics. »

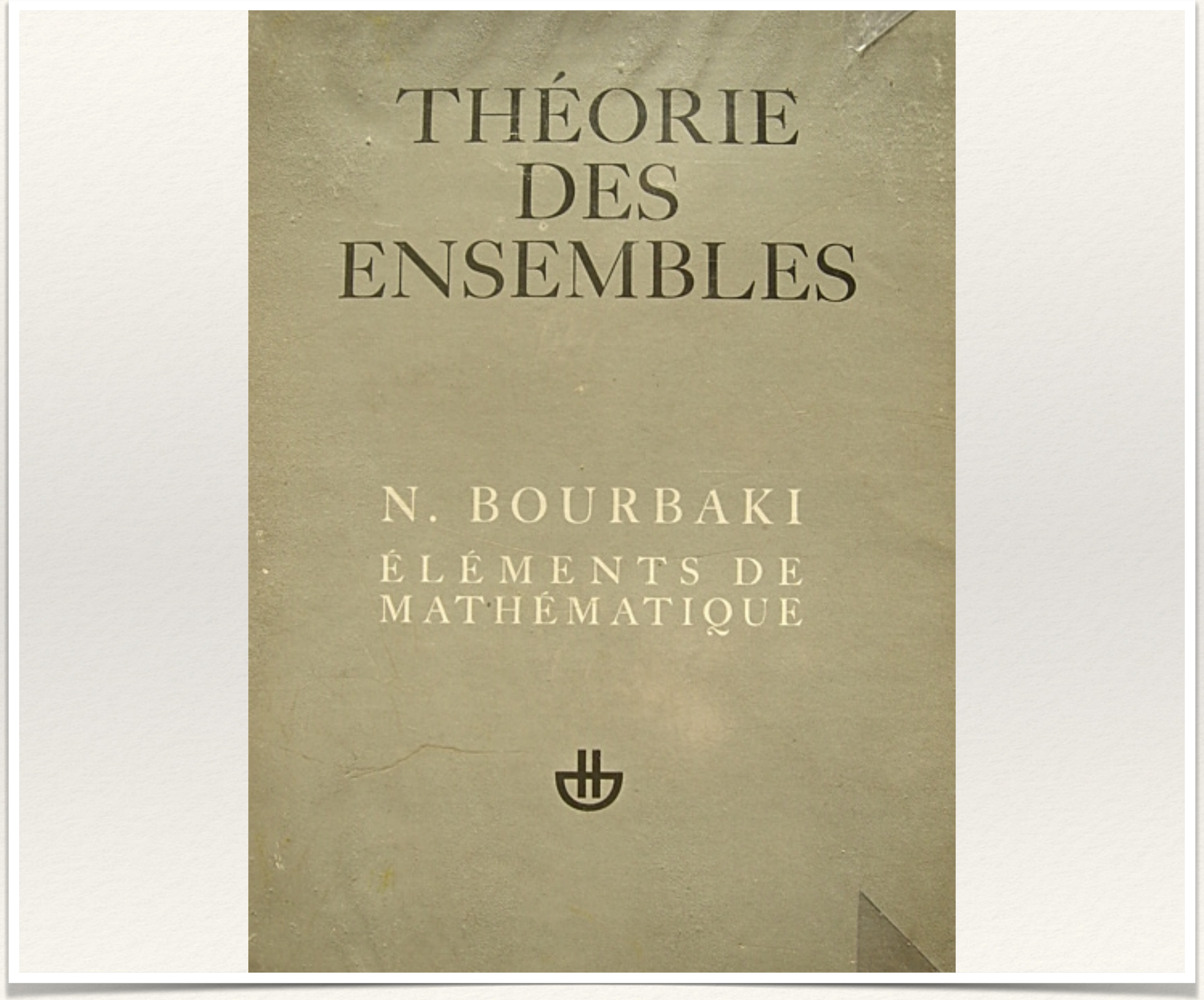
- Bourbaki, The Architecture of Mathematics, 1950.

Abstract structures and the abstract method

- Notice the historical reference in the preceding quote.
- Bourbaki says so explicitly at various places: mathematical structures are *abstracted* from previous mathematical contexts.
- The axiomatic method plays a specific role in the abstraction process: the axioms capture the essential properties of a species of structure.

Bourbaki and categories

- All the structures in Bourbaki's ontology are set-based.
- Categories, functors and natural transformations were introduced by Eilenberg and Mac Lane in (1945). (The concepts were mentioned already in 1942.)
- Members of Bourbaki learns and *uses* categories, functors and natural transformations from early on: Cartan collaborates with Eilenberg from the late 1940s and early 1950s.



Bourbaki and categories

« The method of functors and categories is in some sort of « competition » with the method of structures as developed at present. Unless this « competition » is resolved only one of these methods should be presented at the early stage. Bourbaki is committed by structures for all the material of part I at least. » (Eilenberg, quoted by Krömer)

To recapitulate

- Bourbaki's *Elements* are *not* textbooks. Nor are they research monographs.
- Bourbaki's goal is not to solve problems, it is not to build new theories.
- Bourbaki wants to develop mathematics (at least analysis) so that all the theorems are about *abstract structures* and *combinations* of abstract structures and all the relations that are expressible in these theorems are *transportable*.

In Octobre 1951, Bourbaki writes a poem about functors

The Functors

Reviled be you Cartan for your too long journey
And you too Sammy who lose your fleece.
Better to spare, I believe, your reason
And leave there these games not yet of your age
When will I see you, alas, of a wiser memory
Trying to make a name for yourself? And when will we be able
Of a monster so obvious refuse the impression?
Many newspapers, for sure, would take advantage of it.
Rather the paradise of Cantor, of the elders,
Than your superb work with its bold front.
More than the pure axiom pleases me the fine astuteness.
More the Chinese lemma than your vain article.
More my small Lainé than this chapter twenty.
And more than a satellite a good affine space.

Les Foncteurs.

Honni sois-tu Cartan pour ton trop long voyage,
Et toi aussi Sammy qui perds de ta toison.
Mieux vaudrait ménager, je crois, votre raison
Et laisser là ces jeux pas encor de votre âge.

Quand vous verrai-je, hélas, d'un mémoire plus sage
Tenter la renommée ? Et quand donc pourra-t-on
D'un monstre si patent refuser l'impression ?
Maint journal, à coup sûr, y prendrait avantage.

Plutôt le paradis de Cantor, des aïeux,
Que votre oeuvre superbe au front audacieux.
Plus que l'axiome pûr me plait l'astuce fine.

Plus le lemme chinois que votre article vain.
Plus mon petit Lainé que ce chapitre vingt.
Et plus qu'un satellite un bon espace affine.

Bourbaki, Grothendieck and categories

- Grothendieck studied and worked with Bourbaki's members from the very beginning (Cartan, Dieudonné, Schwartz and, of course, Serre).
- Grothendieck is a guinea pig in February 1950, when Bourbaki held its meeting in Nancy. Weil did not attend that meeting. The same happens the next year.
- In 1952, Grothendieck is present as a guess (not a guinea pig).

Bourbaki, Grothendieck and categories

- At the 1952 meeting, one reads that Cartan is « accused of being unconsciously of bad faith and a verse (alexandrin) captures his mistakes: « qui sème le foncteur récolte la structure » (« who sows the functor reaps the structure »).
- Grothendieck is there to discuss the volume on topological vector spaces…
- Another joke: logicians told Grothendieck that even if all empty sets are equal, some are more equal than others. Grothendieck became berserk and left for Nancy…

Bourbaki, Grothendieck and categories

- In June 1954, Bourbaki decides that Grothendieck will officially become a member. (Schwartz will talk to Grothendieck.)
- At the meeting of February-March 1955, one reads « the structures are broken down » (« Les structures sont en pannes. »)
- At the next meeting (in Chicago, June 1955), there is a lot of discussion about algebraic geometry. Bourbaki is basically stuck.
- At the fall meeting (October 1955), Grothendieck is officially a member and has to write something about sheaves homology.

The plan falls apart in 1956…

- 1956-06-24: Chap. IV (structures) — « An article by Cartier shows that Samuel's results on inductive limits are special cases of ultra general stuff on commutativity of universal problems. This stuff can be stated clearly in the context of categories and functors. Cartier proposes a metamathematical method to introduce the latter without modifying our logical system. But this system is vomited [sic], since it turns its back to the extensional point of view, and above all because with it, it often becomes impossible, without making considerable mental efforts, to tell whether we are doing mathematics, metamathematics, or even metametamathematics. »

Everything falls apart in 1956...

- « It is therefore decided that it is preferable to enlarge the system to bring in categories; at first sight, Gödel's system seems to be appropriate. In order to avoir to have the butt between two chairs [a french expression…], and also not to postpone the publication of a chapter on which we have worked a lot, we decide (in spite of Dixmier's veto, withdrawn in extremis) to send chap. IV to print […]. As for categories and functors, we are finally convinced that it is very important. »
- « Chap V. (Categories and functors) — To start with, Grothendieck will write a kind of leaflet of results in a naive style, so that Bourbaki realizes what it is useful to be able to do. We will then formalize. »

And yet, everything starts in 1956…

- 1956-06-24: Bourbaki is still stuck: « For the last time, perhaps, we heard one of the participants (« congressistes » in French) shouting, while the specialists were arguing about the notion of variety : 'it is as if atheists attended disputes of theologians'. It is decided that it is urgent to put some order in the terminology. »
- Grothendieck is asked to write the abstract chapter, on what that various kinds of varieties have in common from an abstract point of view.

1957-03-17: Congrès du foncteur inflexible

- « …the book on varieties now requires categorical preliminaries, algebraic and analytic which do not appear in the existing books. Roughly: functors, categories [sic], sheaves and fibered spaces [« fibrés »], inductive limits, Lie algebras, nuclear spaces (…), — and all the algebra which is indispensable to operate Grothendieck's machinery. »

1958-06-25: Congrès des hyperplans

- « Book on algebraic topology.

  Sammy makes the speech (« laius » in French) …. Since homological algebra is still being generalized, it is decided to wait before writing it. […] Sammy then tries to sell a carpet [sic] on the logical foundations of categories and functors. *La tribu* does not reproduce it, since it was vomited [sic]. We are waiting for a paper from Lacombe, who was consulted by Serre and Dixmier on the advice of logicians. »

- At that point, Grothendieck is expected to produce a report on algebraic geometry with Dieudonné.

1959-03-07:Congrès « chez mon cousin »

- « Categories —

  The wonderful theory of universes, acclaimed by all, *Cartan dissenting* [english in the original], now allows us to write categories in a convenient mathematical framework. During the Congress, the theory of universes has been enriched with pleasant complements, […] that our readers will find in a short paper entitled « Univers, ensembles artiniens et cardinaux inaccessibles. »

1959-06-25: Congrès du cerceau

- « Categories —

  We have the framework of universes, and Grothendieck has shown during the Congress that any abelian category admits a faithful and exact functor in a category of abelian groups. »

- « I learn that Grothendieck is no longer a member of Bourbaki. (…) It is a scandal that Bourbaki, not only is he not at the forefront of the functorial movement, but he is not even at the tail end… If some of the founding fathers (e.g. Weil) want to reconsider to refrain from influencing Bourbaki in the direction he wants to take, they should say so explicitly. » (La Tribu 53, 1961?, Serge Lang?)

Quoted from Krömer

Categories, functors, etc. are parts of mathematics

- « It is certain that categories, functors, homomorphisms of functors, etc., ... ought to be considered as mathematical objects, that one can freely quantify over and that can be considered elements of sets. This is necessary for two reasons: to apply without constraints the usual mathematical modes of reasoning to functors (...); many mathematical structures are naturally expressed ... as functors. » (quoted by Krömer)

Grothendieck's work

- Grothendieck's work is always *goal oriented:* from Dieudonné-Schwartz's problems to Weil conjectures, Grothendieck's aims is to solve these problems.
- Of course, his method is idiosyncratic: the method of the *rising sea.*
- *Methodological principle: every problem has a proper context, and when it is put in that context, its solution follows from a series of straightforward steps.*

Grothendieck and the functorial language

- From very early on, Grothendieck talks about the *functorial language* and not about category *theory* or the *theory* of functors.
- « The first four chapters contain merely the first definitions concerning general fiber spaces, sheaves, fibre spaces with composition law (including sheaves of groups) and fiber spaces with structure sheaf. *The functor aspect of the notions dealt with has been stressed throughout, and as it now appears should have been stressed even more.* As the proofs of most of the facts stated reduce of course to straightforward verifications, they are only sketched or even omitted, the important point being merely a consistent order in the statement of the main facts. » (1955, *A General Theory of Fiber Spaces with Structure Sheaf*, pp. 1- 2)

The functorial language

- In contrast with Mac Lane 1950, Eilenberg & Steenrood 1952, Cartan & Eilenberg 1954 and Buchsbaum 1954, Grothendieck fully develops a *language, the functorial language,* to solve his problems.
- Like any proper language, it has its own grammar, it allows for certain expressions and the formulation of certain facts.
- Most importantly, it is seen by him, from the very beginning, as an abstraction from the set theoretical language, i.e. any concept expressible in the set theoretical language can be expressed in the functorial language. *And it is more expressive*.

Grothendieck's *languages*

- In fact, Grothendieck talks about many different languages at different times.
- There is the functorial language, the language of schemes, the language of toposes, etc.
- These languages are developed from the functorial language. These latter languages all have, at least at first, specific purposes (e.g. schemes for algebraic geometry, toposes for topology and cohomology theories, etc.).

The functorial language

- As a language, it allows for the *creation* of new mathematical objects and the statement of new mathematical facts.
- *All* of Bourbaki's structures are captured in the functorial language.
- Genuinely new structures are also introduced.
- But this not to say that anything goes (e.g. the creations are not arbitrary) or that mathematical objects thus obtained are *fictions*.

The functorial language

- Given a language (in an informal sense here, not in the logical sense of a formal system), the combinatorial possibilities inherent to its grammar allow us to conjure various kinds of fictitious entities.
- The functorial language encounters its own resistance; not any combinatorial composition of categorical properties gives rise to mathematical objects.
- A great mathematician like Grothendieck not only builds the language, but also knows what are the fruitful, proper, useful, (take your pick), combinations.

The functorial language

- Once the *grammar* of the language has been clarified, one still has to determine what *makes sense* and what does *not* in that language.
- Furthermore, given the level of abstraction of the language developed, reference is a complicated business and completely different from what one finds in set theory, for instance.

Grothendieck and the emergence of mathematical structures

- Grothendieck does not use the abstract method (in contrast with Bourbaki).
- He does not start with a variety of examples to abstract their common properties which are then presented as axioms.
- As is well-known, Grothendieck rarely gives examples or work with examples.
- Claim: Grothendieck uses the functorial language to *design* mathematical structures that are extraordinarily flexible.

On (conceptual) design

- « Design could be viewed as an activity that translates an idea into a blueprint for something useful… » (Design council)
- « It is based on the idea that every design problem begins with an effort to achieve fitness between two entities: the form in question and its context. The form is the solution to the problem; the context defines the problem. In other words, the real object of design is not the form alone, but the ensemble comprising the form and its context. » (Alexander 1964, 14-15)

On (conceptual) design

- A good design provides a good fit between the *function (role, purpose)* of the object designed, the user and the context. It makes sense to talk about a good design, a beautiful design, or even the proper design. (But it is always relative to a user and a context…)
- In designing, one starts with *prototypes* and sometimes it is necessary to step back and start again.
- In his search for a Weil cohomology, Grothendieck illustrates this process (starting with Serre's work and suggestions).

Grothendieck and conceptual design

- Grothendieck does not design *singular objects*, but rather abstract objects that are such that, if one specifies certain parameters, one falls back on well known structures that can be very different form one another.
- The beauty and surprise is that one can prove mathematically significant facts about these designed abstract objects.
- The terminology itself reflects the flexible nature of these objects, e.g. schemes (or schemas), toposes, etc.

Grothendieck and conceptual design

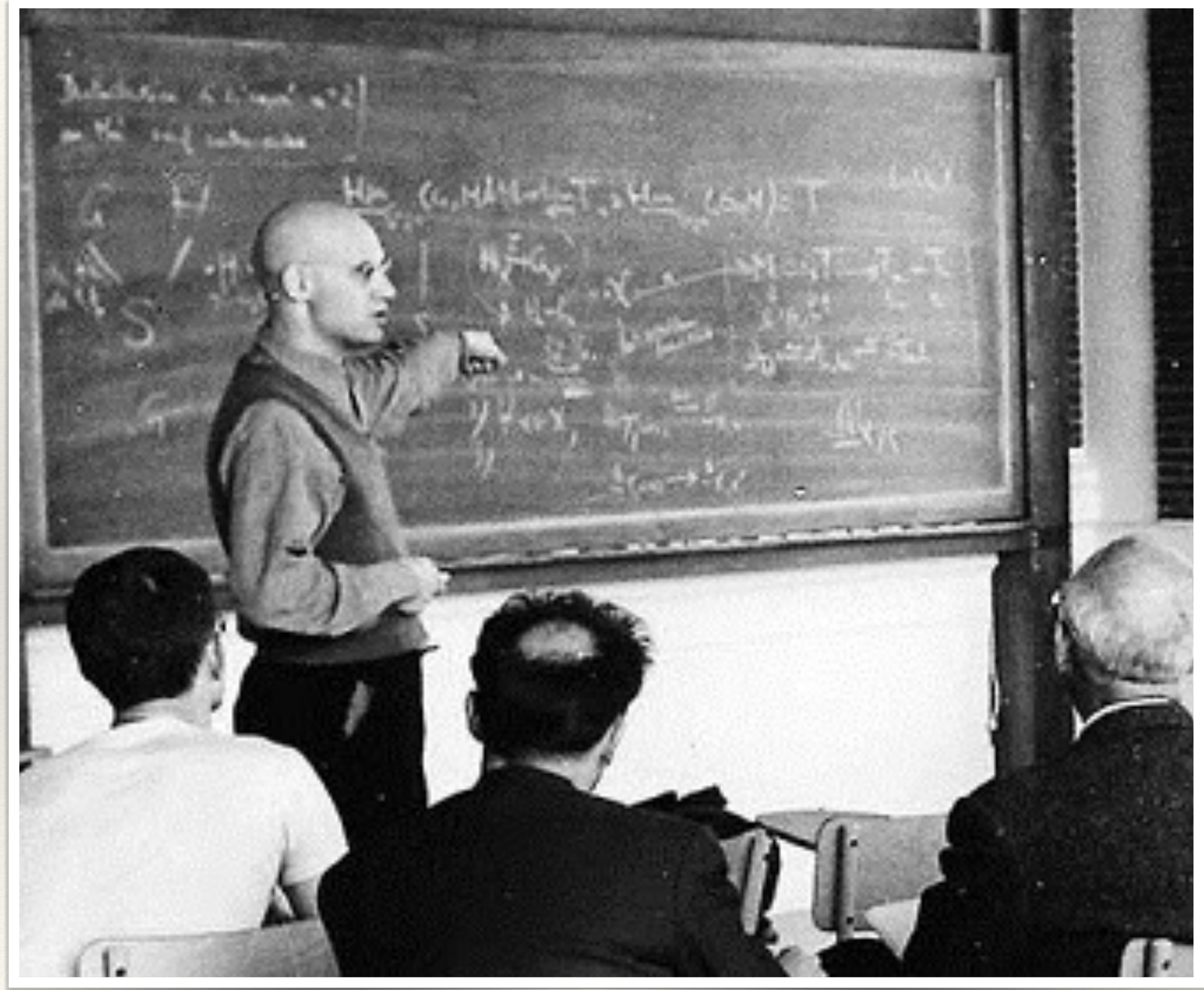
- Examples of this process abound:
  - Abelian categories: this is an extraordinarily elegant axiomatic presentation. Grothendieck's presentation is radically different from Buchsbaum's approach (and Mac Lane).
  - Schemes: this is not given in an axiomatic fashion.
  - Derived categories and Triangulated categories (with Verdier).
  - The most interesting case (in my mind): toposes.

Grothendieck toposes

- Definition: a *topos* is a category $\mathscr{E}$ if there exists a site $\mathscr{C}$ such that $\mathscr{E}$ is equivalent to the category $Set^{\mathscr{C}}$ of sheaves on $\mathscr{C}$.
- This is *not* an axiomatic definition (of course, there is Giraud's theorem, but it is almost an afterthought).
- But it *is* an *operational* definition, and it comes with a formal criterion of identity, i.e. it is given up to an equivalence of categories.
- Nothing is abstracted here. But to the extent that one understands the underlying language, one knows how to proceed.

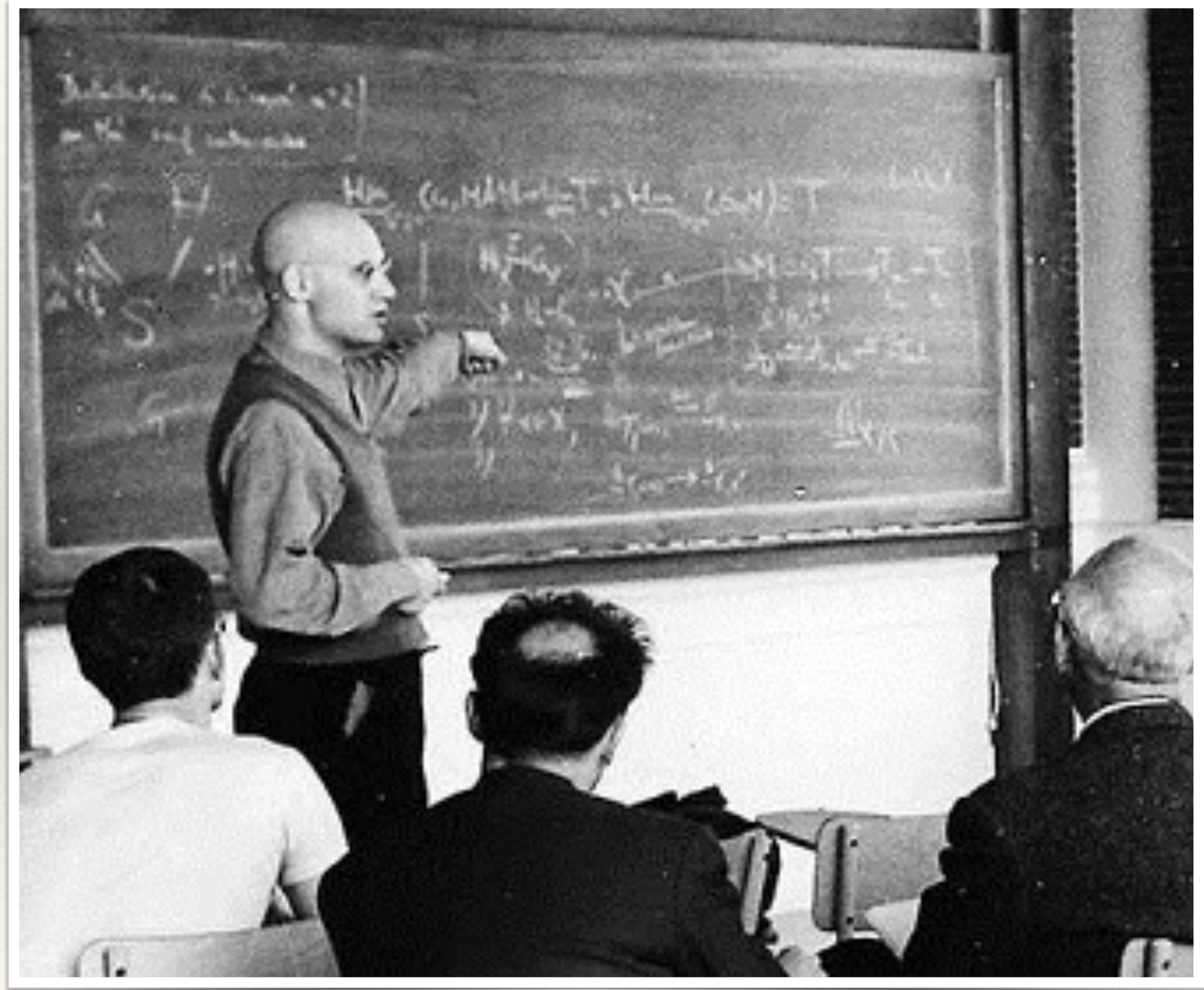
Conclusion

- Bourbaki's abstract mathematical structuralism rests upon a *metamathematical principle*. It is *built in* the grammar of theories.
- In the end, mathematics is developed *up to isomorphism* in Bourbaki's framework.
- Canonical maps are everywhere.



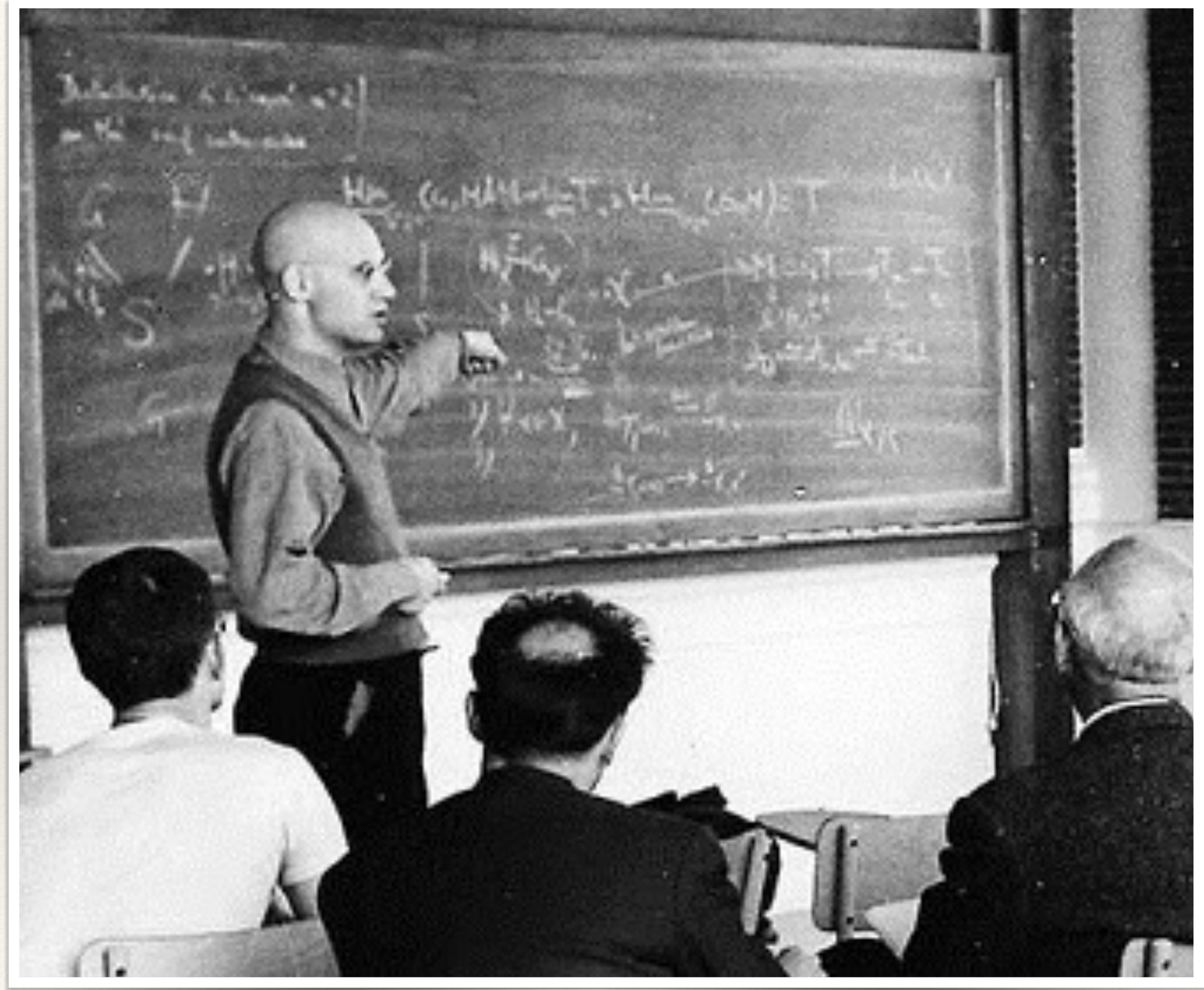
Conclusion

- Bourbaki shows how classical results can be obtained from the combination of abstract structures (sometimes referred to as the « mother structures »).
- Grothendieck shows how (many) results obtained by Bourbaki (and bourbakists) can be obtained from the adoption of the right language and setting certain parameters (the latter being Bourbaki's abstract structures).



Conclusion

- Grothendieck does not specify a general metamathematical framework with respect to structures (Universes are not meant for that).
- More importantly, when one moves to categories themselves, the right notion of isomorphism is the notion of equivalence of categories (and so on for (weak) 2-categories, …, *n*-categories).



Conclusion

- Grothendieck introduces new levels of abstract structures and works up to « isomorphism » all the time.
- They are all fundamental (revolutionary?) changes: new languages, new criteria of identity, new methods of definition, new methods of proof, new ways of dealing with sense and reference, in other words entirely new mathematics.