

Grothendieck fibrations *or* **When aesthetics drives mathematics**

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Grothendieck, a multifarious giant
Mathematics, Logic and Philosophy



CHAPMAN UNIVERSITY

Fibrations are categories varying over a category

Definition

A **fibration** is a functor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

i.e. data as follows

1. for every object b in \mathcal{B} a category $P(b)$
2. for every $f: b' \rightarrow b$ in \mathcal{B} a functor $P(f): P(b) \rightarrow P(b')$

3. for every object b in \mathcal{B} a commutative diagram
$$P(b) \begin{array}{c} \xrightarrow{\text{Id}_{P(b)}} \\ \parallel \\ \xrightarrow{P(\text{id}_b)} \end{array} P(b)$$

4. for every composable pair $g: b'' \rightarrow b', f: b' \rightarrow b$ in \mathcal{B} a commutative diagram

$$\begin{array}{ccccc} & & P(b') & & \\ & \nearrow^{P(f)} & & \searrow^{P(g)} & \\ P(b) & & & & P(b'') \\ & \xrightarrow{P(f \circ g)} & & & \end{array}$$

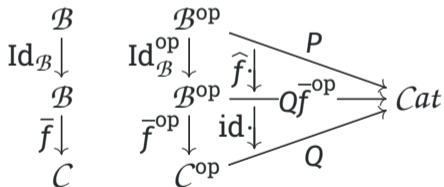
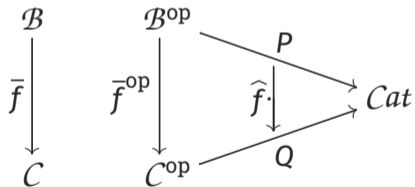
\parallel

Fibrations are categories varying over a category

Definition

A *fibration* is a functor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

A *homomorphism* of fibration is a pair



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Examples

$$\text{Set}^{\text{op}} \xrightarrow{\wp} \text{Pos} \hookrightarrow \text{Cat}$$

$$\text{Ctx}_{\mathcal{L}}^{\text{op}} \xrightarrow{\text{LT}^{\mathcal{T}}} \text{Preord} \hookrightarrow \text{Cat} \quad \text{for } \mathcal{T} \text{ a theory in first order logic}$$

$$\mathcal{G}^{\text{op}} \xrightarrow{\xi} \text{Aut}(\mathcal{H}) \hookrightarrow \text{Cat}$$

$$\text{Set}^{\text{op}} \xrightarrow{\mathcal{C}^{(-)}} \text{Cat}$$

$$\mathcal{B}^{\text{op}} \xrightarrow{F} \text{Set} \hookrightarrow \text{Cat}$$

$$\mathcal{B}^{\text{op}} \xrightarrow{\mathcal{B}/(-)} \text{Cat} \quad \text{when } \mathcal{B} \text{ has pullbacks}$$

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$$\begin{array}{ccc} \text{Set}^{\text{op}} & \xrightarrow{\wp} & \text{Cat} \\ S & \longmapsto & \wp(S) \\ f \downarrow & \longmapsto & \downarrow f^{-1} \\ S' & \longmapsto & \wp(S') \end{array}$$

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$$\begin{array}{ccc} \text{Ctx}_{\mathcal{L}}^{\text{op}} & \xrightarrow{\text{LT}^{\mathcal{T}}} & \text{Cat} \\ (x_1, \dots, x_n) \vdash & \longrightarrow & \text{LT}_{x_1, \dots, x_n} \\ \downarrow & & \downarrow \\ (t_1, \dots, t_n) \vdash & \longrightarrow & -[\tilde{t}/\tilde{x}] \\ \downarrow & & \downarrow \\ (y_1, \dots, y_m) \vdash & \longrightarrow & \text{LT}_{y_1, \dots, y_m} \end{array}$$

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$$\begin{array}{ccc} \text{Set}^{\text{op}} & \xrightarrow{\mathcal{C}^{(-)}} & \text{Cat} \\ S & \longmapsto & \mathcal{C}^S \\ f \downarrow & \longmapsto & \downarrow - \circ f \\ S' & \longmapsto & \mathcal{C}^{S'} \end{array}$$

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$$\begin{array}{ccc} \mathcal{B}^{\text{op}} & \xrightarrow{F} & \text{Cat} \\ b \vdash & \longrightarrow & F(b) \\ f \downarrow & \longrightarrow & \downarrow F(f) \\ b' \vdash & \longrightarrow & F(b') \end{array}$$

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$$\begin{array}{ccc} \mathcal{B}^{\text{op}} & \xrightarrow{\mathcal{B}/(-)} & \text{Cat} \\ b & \longmapsto & \mathcal{B}/b \\ f \downarrow & \longmapsto & \downarrow f^* \\ b' & \longmapsto & \mathcal{B}/b' \end{array}$$

Fibrations are categories varying over a category

Definition

A **fibration** is a **pseudofunctor** $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

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i.e. data as follows

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2. for every $f: b' \rightarrow b$ in \mathcal{B} a functor $P(f): P(b) \rightarrow P(b')$

3. for every object b in \mathcal{B} a natural isomorphism

$$P(b) \begin{array}{c} \xrightarrow{\text{Id}_{P(b)}} \\ \nu_b \downarrow \wr \\ \xrightarrow{P(\text{id}_b)} \end{array} P(b)$$

4. for every composable pair $g: b'' \rightarrow b', f: b' \rightarrow b$ in \mathcal{B} a natural isomorphism

$$P(b) \begin{array}{ccc} & \xrightarrow{P(f)} & P(b') \\ & & \searrow P(g) \\ \xrightarrow{P(f \circ g)} & & P(b'') \end{array} \mu_{g,f} \downarrow \wr$$

which satisfy...

Fibrations are categories varying over a category

Definition

A **fibration** is a pseudofunctor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$
i.e. data as follows... which satisfy
for $f: b' \rightarrow b$

$$\begin{array}{ccc} P(b) & & \\ \text{Id}_{P(b)} \downarrow \begin{array}{c} \xrightarrow{\nu_b} \\ \downarrow \end{array} & & \\ P(b) & \xrightarrow{P(f)} & P(b') \end{array} \begin{array}{c} \nearrow P(f) \\ \nearrow \mu_{f, \text{id}_b} \end{array} \begin{array}{c} P(b) \\ \downarrow \\ P(\text{id}_b) \end{array}$$
$$= \begin{array}{ccc} P(b) & & \\ \text{Id}_{P(b)} \downarrow & & \\ P(b) & \xrightarrow{P(f)} & P(b') \end{array} \begin{array}{c} \nearrow P(f) \\ \nearrow \text{id}_{P(f)} \end{array} \begin{array}{c} P(b) \\ \downarrow \\ P(\text{id}_b) \end{array}$$

and...

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i.e. data as follows... which satisfy
for $f: b' \rightarrow b$...

$$\begin{array}{ccc} P(b) & \xrightarrow{P(f)} & P(b') \\ & \searrow & \downarrow P(\text{id}_{b'}) \\ & & \mathcal{V}_{b'} \\ & & \leftarrow \text{Id}_{P(b')} \\ & & \downarrow \\ & & P(b') \end{array} \quad \begin{array}{c} \mu_{\text{id}_{b'}, f} \\ \downarrow \\ \text{Id}_{P(b')} \end{array} \quad = \quad \begin{array}{ccc} P(b) & \xrightarrow{P(f)} & P(b') \\ & \searrow & \downarrow \text{Id}_{P(b')} \\ & & P(b') \\ & \swarrow \text{id}_{P(f)} & \\ & & P(b') \end{array}$$

and...

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A **fibration** is a pseudofunctor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$
 i.e. data as follows... which satisfy... and
 for $h: b''' \rightarrow b''$, $g: b'' \rightarrow b'$, $f: b' \rightarrow b$

$$\begin{array}{ccc}
 P(b) & \xrightarrow{P(f \circ g \circ h)} & P(b''') \\
 \downarrow P(f) & \nearrow P(g \circ h) & \uparrow P(h) \\
 P(b') & \xrightarrow{P(g)} & P(b'')
 \end{array}
 \quad
 \begin{array}{ccc}
 P(b) & \xrightarrow{P(f \circ g \circ h)} & P(b''') \\
 \downarrow P(f) & \searrow P(f \circ g) & \uparrow P(h) \\
 P(b') & \xrightarrow{P(g)} & P(b'')
 \end{array}$$

The diagram shows two commutative triangles representing the naturality of the pseudofunctor P . The left triangle has vertices $P(b)$, $P(b')$, and $P(b'')$ with edges $P(f)$, $P(g)$, and $P(g \circ h)$. The right triangle has vertices $P(b)$, $P(b')$, and $P(b'')$ with edges $P(f)$, $P(g)$, and $P(f \circ g)$. The top edge of both triangles is $P(f \circ g \circ h)$. The bottom edge of the left triangle is $P(g)$, and the bottom edge of the right triangle is $P(g)$. The right edge of the left triangle is $P(h)$, and the right edge of the right triangle is $P(h)$. The left edge of the left triangle is $P(f)$, and the left edge of the right triangle is $P(f)$. The diagonal edge of the left triangle is $P(g \circ h)$, and the diagonal edge of the right triangle is $P(f \circ g)$. The 2-cells $\mu_{g \circ h, f}$ and $\mu_{h, g}$ are shown as arrows from $P(g \circ h)$ to $P(f \circ g \circ h)$ and from $P(h)$ to $P(f \circ g \circ h)$ respectively. The 2-cells $\mu_{h, f \circ g}$ and $\mu_{g, f}$ are shown as arrows from $P(h)$ to $P(f \circ g \circ h)$ and from $P(g)$ to $P(f \circ g \circ h)$ respectively. The two triangles are separated by an equals sign, indicating that the two diagrams are equivalent.

Fibrations are categories varying over a category, II

Definition

A *fibration* is a functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$ such that for every e and f

$$\begin{array}{ccc} & e & \\ & \downarrow & \\ b' & \xrightarrow{f} & b \\ & & \downarrow p \\ & & \mathcal{B} \end{array}$$

Fibrations are categories varying over a category, II

Definition

A *fibration* is a functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$ such that for every e and f there is \widehat{f}

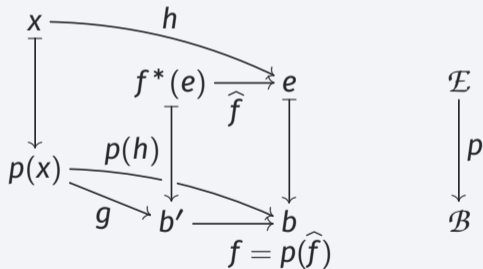
$$\begin{array}{ccc} f^*(e) & \xrightarrow{\widehat{f}} & e \\ \downarrow & & \downarrow \\ b' & \xrightarrow{f} & b \\ & f = p(\widehat{f}) & \end{array} \qquad \begin{array}{c} \mathcal{E} \\ \downarrow p \\ \mathcal{B} \end{array}$$

Fibrations are categories varying over a category, II

Definition

A *fibration* is a functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$ such that

for every e and f there is \widehat{f} universal with the property

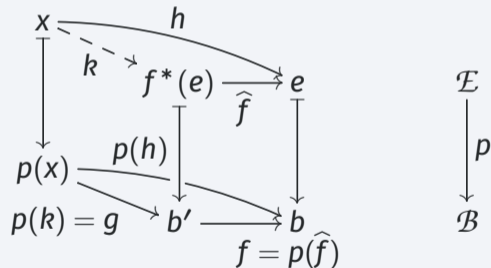


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Comparing pseudofunctors and fibrations

Pseudofunctors

Fibrations

$$\mathcal{B}^{\text{op}} \xrightarrow{p^{-1}} \mathit{Cat} \longleftarrow \mathcal{E} \xrightarrow{p} \mathcal{B}$$

the category $p^{-1}(b)$:

$$\begin{array}{c} e \quad \text{s.t. } p(e) = b \\ \downarrow v \\ e' \quad \text{s.t. } p(d) = b \end{array} \quad \begin{array}{l} v: e \rightarrow d \text{ in } \mathcal{E} \\ \text{s.t. } p(v) = \text{id}_b \end{array}$$

Comparing pseudofunctors and fibrations

Pseudofunctors

Fibrations

$$\mathcal{B}^{\text{op}} \xrightarrow{p^{-1}} \mathit{Cat} \longleftarrow \mathcal{E} \xrightarrow{p} \mathcal{B}$$

the functor

$$p^{-1}(b) \xrightarrow{p^{-1}(f: b' \rightarrow b)} p^{-1}(b')$$

$$e \longmapsto f^*(e)$$

$$\begin{array}{ccc} f^*(e) & \xrightarrow{\hat{f}} & e \\ \downarrow & & \downarrow \\ b' & \xrightarrow{f = p(\hat{f})} & b \end{array}$$

Comparing pseudofunctors and fibrations

Pseudofunctors

Fibrations

$$\mathcal{B}^{\text{op}} \xrightarrow{p^{-1}} \mathit{Cat} \longleftarrow \mathcal{E} \xrightarrow{p} \mathcal{B}$$

$$\mathcal{B}^{\text{op}} \xrightarrow{P} \mathit{Cat} \longleftarrow \int P \xrightarrow{\text{pr}_1} \mathcal{B}$$

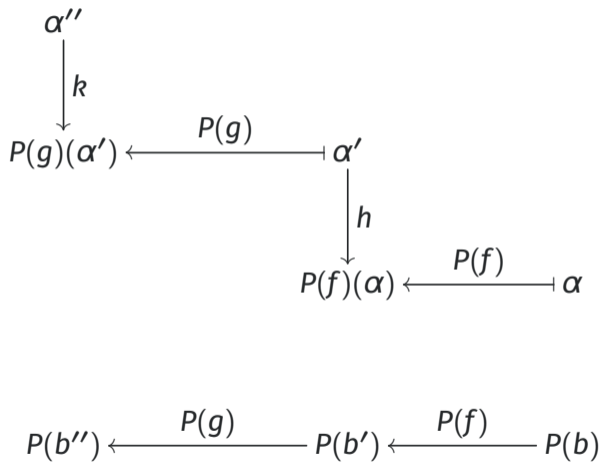
the category $\int P$:

$$(b', \alpha') \xrightarrow[\substack{f: b' \rightarrow b \text{ in } \mathcal{B} \\ h: \alpha \rightarrow P(f)(\alpha) \text{ in } P(b')}]^{(f, h)} (b, \alpha) \text{ s.t. } \alpha \in P(b)$$

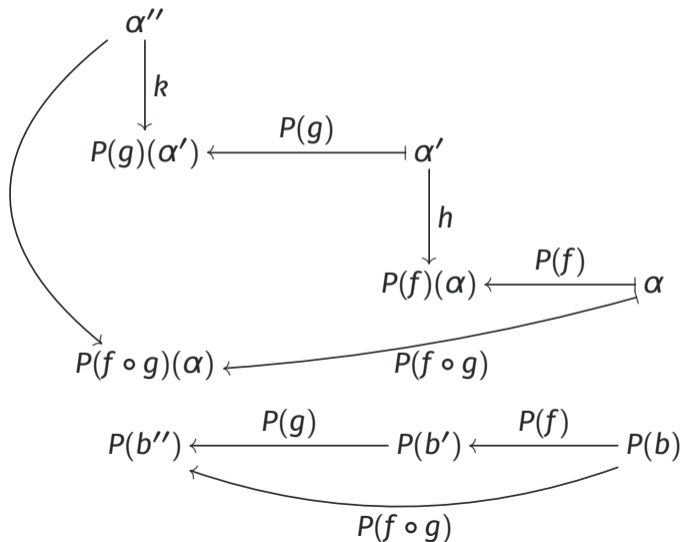
composition is

$$(b'', \alpha'') \xrightarrow{(g, k)} (b', \alpha') \xrightarrow{(f, h)} (b, \alpha) \xrightarrow{(f \circ g, \mu_{g,f} \circ (P(g)(h)) \circ k)} (b, \alpha)$$

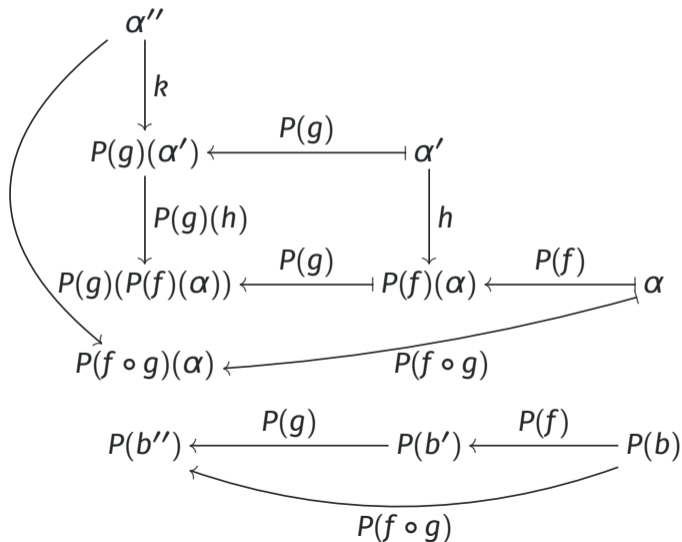
A picture of composition in the category $\int P$



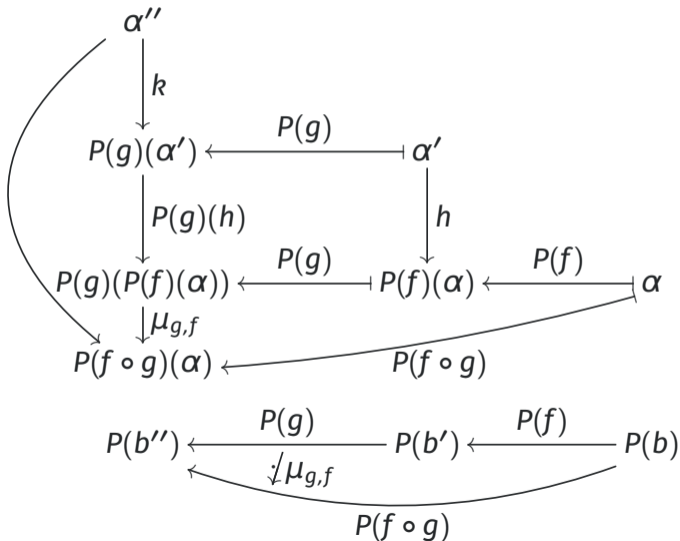
A picture of composition in the category $\int P$



A picture of composition in the category $\int P$



A picture of composition in the category $\int P$



Comparing pseudofunctors and fibrations, II

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\mathcal{B} with pullbacks

Fibration $\int P \xrightarrow{\text{pr}_1} \mathcal{B}$

$$\text{Sub} \longrightarrow \text{Set}$$

$$E \longrightarrow E'$$

$$\begin{array}{ccc} \int & & \int \\ \downarrow f & \dashrightarrow & \downarrow f \\ S & \longrightarrow & S' \end{array}$$

Comparing pseudofunctors and fibrations, II

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Pseudofunctor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

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\mathcal{B} with pullbacks

Fibration $\int P \xrightarrow{\text{pr}_1} \mathcal{B}$

$$\begin{array}{ccc} (\vec{x}, \varphi) & & \\ & \searrow \vec{t} & \\ \text{s.t. } \vec{x} | \varphi \vdash_{\mathcal{T}} \psi[\vec{t}/\vec{x}'] & & \\ & \swarrow & \\ & (\vec{x}', \psi) & \end{array} \quad \longmapsto \quad \vec{x} \xrightarrow{\vec{t}} \vec{x}'$$

Comparing pseudofunctors and fibrations, II

Examples

Pseudofunctor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

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\mathcal{B} with pullbacks

Fibration $\int P \xrightarrow{\text{pr}_1} \mathcal{B}$

$$\mathcal{G} \times_{\xi} \mathcal{H} \xrightarrow{\text{pr}_1} \mathcal{G}$$

$$(g, h) \cdot (g', h') = (gg', \xi_{g'}(h)h')$$

Comparing pseudofunctors and fibrations, II

Examples

Pseudofunctor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

$$\text{Set}^{\text{op}} \xrightarrow{\mathcal{P}} \text{Pos} \hookrightarrow \text{Cat}$$

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\mathcal{B} with pullbacks

Fibration $\int P \xrightarrow{\text{pr}_1} \mathcal{B}$

$$\mathcal{G} \times_{\xi} \mathcal{H} \xrightarrow{\text{pr}_1} \mathcal{G}$$

$$\text{Fam}(\mathcal{C}) \longrightarrow \text{Set}$$

$$\begin{array}{ccc} (c_i)_{i \in S} & \xrightarrow{(r, (f_i)_{i \in S})} & S \xrightarrow{f} S' \\ f_i: c_i \rightarrow c'_{r(i)}, i \in S & \searrow & \\ & & (c'_j)_{j \in S'} \end{array}$$

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\mathcal{B} with pullbacks

Fibration $\int P \xrightarrow{\text{pr}_1} \mathcal{B}$

$$\mathcal{G} \times_{\xi} \mathcal{H} \xrightarrow{\text{pr}_1} \mathcal{G}$$

$$\text{Fam}(C) \xrightarrow{\quad\quad\quad} \text{Set}$$

$\int F \xrightarrow{\text{pr}_1} \mathcal{B}$ is a **discrete** fibration

Comparing pseudofunctors and fibrations, II

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Pseudofunctor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

$$\text{Set}^{\text{op}} \xrightarrow{\mathcal{P}} \text{Pos} \hookrightarrow \text{Cat}$$

$$\text{Ctx}_{\mathcal{L}}^{\text{op}} \xrightarrow{\text{LT}^{\mathcal{T}}} \text{Preord} \hookrightarrow \text{Cat}$$

for \mathcal{T} a theory in first order logic

$$\mathcal{G}^{\text{op}} \xrightarrow{\xi} \text{Aut}(\mathcal{H}) \hookrightarrow \text{Cat}$$

$$\text{Set}^{\text{op}} \xrightarrow{C^{(-)}} \text{Cat}$$

$$\mathcal{B}^{\text{op}} \xrightarrow{F} \text{Set} \hookrightarrow \text{Cat}$$

$$\mathcal{B}^{\text{op}} \xrightarrow{\mathcal{B}/(-)} \text{Cat}$$

\mathcal{B} with pullbacks

Fibration $\int P \xrightarrow{\text{pr}_1} \mathcal{B}$

$$\mathcal{G} \ltimes_{\xi} \mathcal{H} \xrightarrow{\text{pr}_1} \mathcal{G}$$

$$\text{Fam}(C) \xrightarrow{\quad\quad\quad} \text{Set}$$

$\int F \xrightarrow{\text{pr}_1} \mathcal{B}$ is a *discrete* fibration

$$\mathcal{B}^2 \xrightarrow{\text{cod}} \mathcal{B}$$

$$2 = \boxed{\bullet \longrightarrow *}$$

Split fibrations

Definition

A fibration $\mathcal{D} \xrightarrow{S} \mathcal{B}$ is *split* when it is of the form $\int P \xrightarrow{\text{pr}_1} \mathcal{B}$ for some functor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

$$\text{SplFibr}(\mathcal{B}) = \text{cat}([\mathcal{B}^{\text{op}}, \text{Set}])$$

Examples

$$\text{Sub} \longrightarrow \text{Set}$$

$$\mathcal{G} \times_{\xi} \mathcal{H} \xrightarrow{\text{pr}_1} \mathcal{G}$$

$$\text{Fam}(\mathcal{C}) \longrightarrow \text{Set}$$

$$\text{Discrete fibrations } \int F \xrightarrow{\text{pr}_1} \mathcal{B}$$

Split fibrations

Definition

A fibration $\mathcal{D} \xrightarrow{S} \mathcal{B}$ is *split* when it is of the form $\int P \xrightarrow{\text{pr}_1} \mathcal{B}$ for some functor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

$$\text{SplFibr}(\mathcal{B}) = \text{cat}([\mathcal{B}^{\text{op}}, \text{Set}])$$

Examples

Representable discrete fibrations

$$\begin{array}{c} \mathcal{B}/a \\ \downarrow \mathcal{L}_a \\ \mathcal{B} \end{array}$$

$$\begin{array}{ccc} & & b \\ & \swarrow & \\ a & \xleftarrow{e} & \\ & \searrow & \\ & & b \end{array}$$

Split fibrations

Definition

A fibration $\mathcal{D} \xrightarrow{S} \mathcal{B}$ is *split* when it is of the form $\int P \xrightarrow{\text{pr}_1} \mathcal{B}$ for some functor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

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Examples

Representable discrete fibrations

$$\begin{array}{c} \mathcal{B}/a \\ \downarrow \wr_a \\ \mathcal{B} \end{array}$$

$$\begin{array}{ccc} & & b \\ & \swarrow & \\ a & \xleftarrow{e} & \\ & \searrow & \\ b' & \xrightarrow{f} & b \end{array}$$

Split fibrations

Definition

A fibration $\mathcal{D} \xrightarrow{S} \mathcal{B}$ is *split* when it is of the form $\int^P \mathbb{P}r_1 \rightarrow \mathcal{B}$ for some functor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

$$\text{SplFibr}(\mathcal{B}) = \text{cat}([\mathcal{B}^{\text{op}}, \text{Set}])$$

Examples

Representable discrete fibrations

$$\begin{array}{c} \mathcal{B}/a \\ \downarrow \mathcal{L}_a \\ \mathcal{B} \end{array}$$

$$\begin{array}{ccccc} & & f & & \\ b' & \xrightarrow{\quad} & & \xrightarrow{\quad} & b \\ & \searrow & & \swarrow & \\ & & a & & \\ & \swarrow & & \searrow & \\ b' & \xrightarrow{\quad} & & \xrightarrow{\quad} & b \\ & & f & & \end{array}$$

Split fibrations

Definition

A fibration $\mathcal{D} \xrightarrow{S} \mathcal{B}$ is *split* when it is of the form $\int P \xrightarrow{\text{pr}_1} \mathcal{B}$ for some functor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

$$\text{SplFibr}(\mathcal{B}) = \text{cat}([\mathcal{B}^{\text{op}}, \text{Set}])$$

Examples

The Yoneda fibration

$$\begin{array}{c} \mathcal{B}_1 \\ \downarrow \text{y} \\ \mathcal{B}^{\text{op}} \times \mathcal{B} \end{array}$$

$$\begin{array}{ccccc} & & b' & \xrightarrow{f_2} & b \\ & f_1 e f_2 \swarrow & & & \searrow e \\ a' & \xleftarrow{\tau} & a & & \\ & \searrow f_1 & & & \searrow \\ & (a', b') & \xrightarrow{(f_1, f_2)} & (a, b) & \end{array}$$

Split fibrations

Definition

A fibration $\mathcal{D} \xrightarrow{S} \mathcal{B}$ is *split* when it is of the form $\int P \xrightarrow{\text{pr}_1} \mathcal{B}$ for some functor $\mathcal{B}^{\text{op}} \xrightarrow{P} \text{Cat}$

$$\text{SplFibr}(\mathcal{B}) = \text{cat}([\mathcal{B}^{\text{op}}, \text{Set}])$$

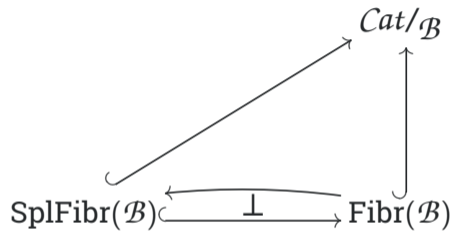
Examples

For C an internal category in \mathcal{B}

$$\int_b \mathcal{K}(b, C) \xrightarrow{\text{pr}_1} \mathcal{B}$$

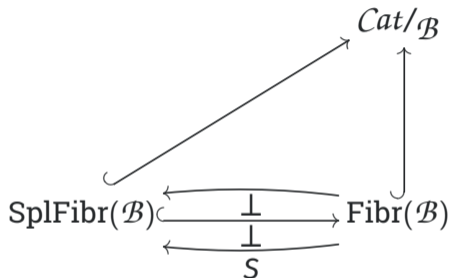
$$\text{Top} \xrightarrow{U} \text{Set}$$

Split fibrations



Split fibrations

$$S(p) = \int_a \text{Fibr}(\mathcal{B})(\mathcal{J}_a, p) \xrightarrow{\text{pr}_1} \mathcal{B}$$



Theorem (Fibered Yoneda Lemma)

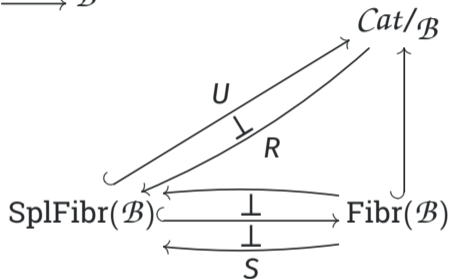
For a fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$ and a split fibration $\mathcal{D} \xrightarrow{s} \mathcal{B}$

$$\text{Fibr}(\mathcal{B})(s, p) \equiv \text{SplFibr}(\mathcal{B})(s, S(p))$$

Split fibrations

$$S(p) = \int_a \text{Fibr}(\mathcal{B})(\mathcal{J}_a, p) \xrightarrow{\text{pr}_1} \mathcal{B}$$

$$R(t) = \int_a \text{Cat}/\mathcal{B}(\mathcal{J}_a, t) \xrightarrow{\text{pr}_1} \mathcal{B}$$

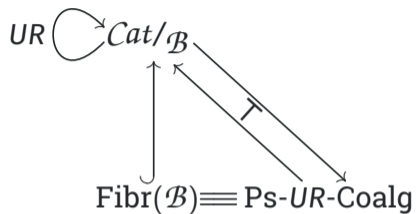


Theorem

For a functor $\mathcal{C} \xrightarrow{t} \mathcal{B}$ and a split fibration $\mathcal{D} \xrightarrow{s} \mathcal{B}$

$$\text{Cat}/\mathcal{B}(U(s), t) \equiv \text{SplFibr}(\mathcal{B})(s, R(t))$$

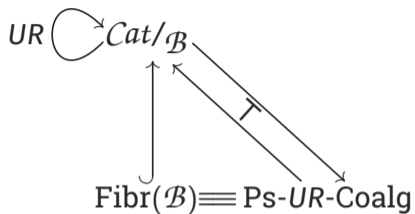
Fibrations are pseudo-coalgebras



Theorem

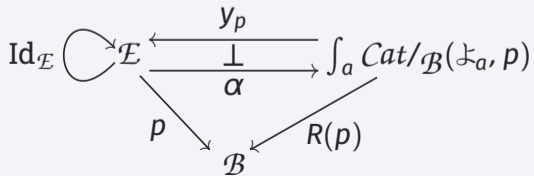
The comonad on UR is KZ and the pseudo-coalgebras are fibrations

Fibrations are pseudo-coalgebras



Corollary

For a fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$ there is a diagram





**Università
di Genova**