Università di Genova | DIMA DIPARTIMENTO DI MATEMATICA

Grothendieck fibrations
or When aestethics drives mathematics

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Grothendieck, a multifarious giant
Mathematics, Logic and Philosophy
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Fibrations are categories varying over a category

Definition

A *fibration* is a functor $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{C}at$
i.e. data as follows

1. for every object b in $\mathcal{B}$ a category $P(b)$
2. for every $f\colon b' \longrightarrow b$ in $\mathcal{B}$ a functor $P(f)\colon P(b) \longrightarrow P(b')$
3. for every object b in $\mathcal{B}$ a commutative diagram $$\begin{tikzcd} P(b) \arrow[r, shift left=2, "\mathrm{Id}_{P(b)}"] \arrow[r, shift right=2, "P(\mathrm{id}_b)"'] \arrow[r, phantom, "\shortparallel"] & P(b) \end{tikzcd}$$
4. for every composable pair $g\colon b'' \longrightarrow b'$, $f\colon b' \longrightarrow b$ in $\mathcal{B}$ a commutative diagram $$\begin{tikzcd} & P(b') \arrow[dr, "P(g)"] \arrow[d, phantom, "\|"] & \\ P(b) \arrow[ur, "P(f)"] \arrow[rr, "P(f \circ g)"'] & {} & P(b'') \end{tikzcd}$$

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A *homomorphism* of fibration is a pair

$$
\begin{tikzcd}
\mathcal{B} \arrow[dd, "\overline{f}"'] & \mathcal{B}^{\mathrm{op}} \arrow[dd, "\overline{f}^{\mathrm{op}}"'] \arrow[drr, "P"] & & \\
& & \arrow[phantom, "\widehat{f}\cdot\Downarrow"] & \mathcal{C}at \\
\mathcal{C} & \mathcal{C}^{\mathrm{op}} \arrow[urr, "Q"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B} \arrow[d, "\mathrm{Id}_{\mathcal{B}}"'] & \mathcal{B}^{\mathrm{op}} \arrow[d, "\mathrm{Id}^{\mathrm{op}}_{\mathcal{B}}"'] \arrow[drr, "P"] & & \\
\mathcal{B} \arrow[d, "\overline{f}"'] & \mathcal{B}^{\mathrm{op}} \arrow[d, "\overline{f}^{\mathrm{op}}"'] \arrow[rr, "Q\overline{f}^{\mathrm{op}}" description] \arrow[r, phantom, shift left=4, "\widehat{f}\cdot\Downarrow"] \arrow[r, phantom, shift right=4, "\mathrm{id}\cdot\Downarrow"] & & \mathcal{C}at \\
\mathcal{C} & \mathcal{C}^{\mathrm{op}} \arrow[urr, "Q"'] & &
\end{tikzcd}
$$

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Definition

A *fibration* is a functor $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{C}at$

Examples

$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\wp} \mathcal{P}os \hookrightarrow \mathcal{C}at$

$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}} \mathcal{P}reord \hookrightarrow \mathcal{C}at$ for $\mathcal{T}$ a theory in first order logic

$\mathcal{G}^{\mathrm{op}} \xrightarrow{\xi} \mathrm{Aut}(\mathcal{H}) \hookrightarrow \mathcal{C}at$

$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\mathcal{C}^{(-)}} \mathcal{C}at$

$\mathcal{B}^{\mathrm{op}} \xrightarrow{F} \mathcal{S}et \hookrightarrow \mathcal{C}at$

$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}/(-)} \mathcal{C}at$ when $\mathcal{B}$ has pullbacks

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Examples

$$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\wp} \mathcal{P}os \hookrightarrow \mathcal{C}at$$

$$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}} \mathcal{P}reord \hookrightarrow \mathcal{C}at$$ for $\mathcal{T}$ a theory in first order logic

$$\mathcal{G}^{\mathrm{op}} \xrightarrow{\xi} \mathrm{Aut}(\mathcal{H}) \hookrightarrow \mathcal{C}at$$

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$$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}/(-)} \mathcal{C}at$$ when $\mathcal{B}$ has pullbacks

$$\begin{tikzcd}
\mathcal{S}et^{\mathrm{op}} \arrow[r, "\wp"] & \mathcal{C}at \\
S \arrow[r, mapsto] \arrow[d, "f"'] & \wp(S) \arrow[d, "f^{-1}"] \\
S' \arrow[r, mapsto] & \wp(S')
\end{tikzcd}$$

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Examples

$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\wp} \mathcal{P}os \hookrightarrow \mathcal{C}at$

$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}} \mathcal{P}reord \hookrightarrow \mathcal{C}at$ for $\mathcal{T}$ a theory in first order logic

$$\begin{tikzcd}
\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \arrow[rr, "\mathrm{LT}^{\mathcal{T}}"] & & \mathcal{C}at \\
(x_1, \ldots, x_n) \arrow[rr, mapsto] \arrow[d, red] & & \mathrm{LT}_{x_1,\ldots,x_n} \arrow[d, "-[\vec{t}/\vec{x}]"] \\
(y_1, \ldots, y_m) \arrow[rr, mapsto] & & \mathrm{LT}_{y_1,\ldots,y_m}
\end{tikzcd}$$

$(t_1, \ldots, t_n) \longmapsto -[\vec{t}/\vec{x}]$

$\mathcal{G}^{\mathrm{op}} \xrightarrow{\xi} \mathrm{Aut}(\mathcal{H}) \hookrightarrow \mathcal{C}at$

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$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\mathcal{C}^{(-)}} \mathcal{C}at$

$$\begin{tikzcd}
\mathcal{S}et^{\mathrm{op}} \arrow[r, "\mathcal{C}^{(-)}"] & \mathcal{C}at \\
S \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{C}^{S} \arrow[d, "- \circ f"] \\
S' \arrow[r, mapsto] & \mathcal{C}^{S'}
\end{tikzcd}$$

$f \longmapsto - \circ f$

$\mathcal{B}^{\mathrm{op}} \xrightarrow{F} \mathcal{S}et \hookrightarrow \mathcal{C}at$

$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}/(-)} \mathcal{C}at$ when $\mathcal{B}$ has pullbacks

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$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\mathcal{C}^{(-)}} \mathcal{C}at$

$\mathcal{B}^{\mathrm{op}} \xrightarrow{F} \mathcal{S}et \hookrightarrow \mathcal{C}at$

$$
\begin{tikzcd}
\mathcal{B}^{\mathrm{op}} \arrow[rr, "F"] & & \mathcal{C}at \\
b \arrow[rr, mapsto] \arrow[d, "f"'] & & F(b) \arrow[d, "F(f)"] \\
b' \arrow[rr, mapsto] & & F(b')
\end{tikzcd}
\qquad f \longmapsto F(f)
$$

$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}/(-)} \mathcal{C}at$ when $\mathcal{B}$ has pullbacks

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$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}/(-)} \mathcal{C}at$ when $\mathcal{B}$ has pullbacks

$$\begin{array}{ccc} \mathcal{B}^{\mathrm{op}} & \xrightarrow{\mathcal{B}/(-)} & \mathcal{C}at \\ b & \longmapsto & \mathcal{B}/_b \\ f \downarrow & \longmapsto & \downarrow f^* \\ b' & \longmapsto & \mathcal{B}/_{b'} \end{array}$$

Fibrations are categories varying over a category

Definition

A *fibration* is a pseudofunctor $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{C}at$

Examples

$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\wp} \mathcal{P}os \hookrightarrow \mathcal{C}at$

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$$\begin{tikzcd}
\mathcal{B}^{\mathrm{op}} \arrow[r, "\mathcal{B}/(-)"] & \mathcal{C}at \\
b \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{B}/_b \arrow[d, "f^*"] \\
b' \arrow[r, mapsto] & \mathcal{B}/_{b'}
\end{tikzcd}$$

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1. for every object b in $\mathcal{B}$ a category $P(b)$
2. for every $f \colon b' \longrightarrow b$ in $\mathcal{B}$ a functor $P(f) \colon P(b) \longrightarrow P(b')$
3. for every object b in $\mathcal{B}$ a natural isomorphism $$\begin{tikzcd} P(b) \arrow[r, bend left, "\mathrm{Id}_{P(b)}"] \arrow[r, bend right, "P(\mathrm{id}_b)"'] \arrow[r, phantom, "\nu_b \cdot \Downarrow \wr"] & P(b) \end{tikzcd}$$
4. for every composable pair $g \colon b'' \longrightarrow b'$, $f \colon b' \longrightarrow b$ in $\mathcal{B}$ a natural isomorphism

$$\begin{tikzcd} & P(b') \arrow[dr, "P(g)"] & \\ P(b) \arrow[ur, "P(f)"] \arrow[rr, "P(f \circ g)"'] \arrow[rr, phantom, bend left=15, "\mu_{g,f} \cdot \Downarrow \wr"] & & P(b'') \end{tikzcd}$$

which satisfy…

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i.e. data as follows… which satisfy

for $f\colon b' \longrightarrow b$

$$
\begin{tikzcd}
P(b) \arrow[d, "\mathrm{Id}_{P(b)}"', bend right=20] \arrow[d, "P(\mathrm{id}_b)", bend left=20] \arrow[d, phantom, "\xrightarrow{\nu_b}"] \arrow[drr, "P(f)"] & & \\
P(b) \arrow[rr, "P(f)"'] \arrow[urr, phantom, "\mu_{f,\mathrm{id}_b} \nearrow"] & & P(b')
\end{tikzcd}
=
\begin{tikzcd}
P(b) \arrow[d, "\mathrm{Id}_{P(b)}"'] \arrow[drr, "P(f)"] & & \\
P(b) \arrow[rr, "P(f)"'] \arrow[urr, phantom, "\mathrm{id}_{P(f)} \nearrow"] & & P(b')
\end{tikzcd}
$$

and…

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Definition

A *fibration* is a pseudofunctor $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{C}at$

i.e. data as follows... which satisfy

for $f \colon b' \longrightarrow b$... and

$$
\begin{tikzcd}
P(b) \arrow[r, "P(f)"] \arrow[dr, "P(f)"'] \arrow[dr, phantom, shift left=4ex, "\Leftarrow \mu_{\mathrm{id}_{b'},f}"] & P(b') \arrow[d, shift right=2ex, "P(\mathrm{id}_{b'})"'] \arrow[d, shift left=2ex, "\mathrm{Id}_{P(b')}"] \arrow[d, phantom, "\overset{\nu_{b'}}{\Leftarrow}"] \\
& P(b')
\end{tikzcd}
=
\begin{tikzcd}
P(b) \arrow[r, "P(f)"] \arrow[dr, "P(f)"'] \arrow[dr, phantom, shift left=4ex, "\Leftarrow \mathrm{id}_{P(f)}"] & P(b') \arrow[d, "\mathrm{Id}_{P(b')}"] \\
& P(b')
\end{tikzcd}
$$

and...

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Definition

A *fibration* is a pseudofunctor $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{C}at$
i.e. data as follows… which satisfy… and
for $h\colon b''' \longrightarrow b''$, $g\colon b'' \longrightarrow b'$, $f\colon b' \longrightarrow b$

$$
\begin{tikzcd}
P(b) \arrow[rr, "P(f \circ g \circ h)"] \arrow[d, "P(f)"'] & \arrow[d, phantom, "\mu_{g\circ h,f}\Uparrow"] & P(b''') \\
P(b') \arrow[urr, "P(g \circ h)"] \arrow[rr, "P(g)"'] & \arrow[r, phantom, "\mu_{h,g}\Leftarrow"] & P(b'') \arrow[u, "P(h)"']
\end{tikzcd}
=
\begin{tikzcd}
P(b) \arrow[rr, "P(f \circ g \circ h)"] \arrow[d, "P(f)"'] \arrow[drr, "P(f \circ g)"] & \arrow[r, phantom, "\mu_{h,f\circ g}"] & P(b''') \\
P(b') \arrow[rr, "P(g)"'] & \arrow[u, phantom, "\mu_{g,f}\Nearrow"] & P(b'') \arrow[u, "P(h)"']
\end{tikzcd}
$$

Fibrations are categories varying over a category, II

Definition

A *fibration* is a functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$ such that

for every e and f

$$\begin{tikzcd} & e \arrow[d, mapsto] \\ b' \arrow[r, "f"'] & b \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E} \arrow[d, "p"] \\ \mathcal{B} \end{tikzcd}$$

Fibrations are categories varying over a category, II

Definition

A *fibration* is a functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$ such that
for every e and f there is $\widehat{f}$

$$\begin{tikzcd}
f^*(e) \arrow[r, "\widehat{f}"'] \arrow[d, mapsto] & e \arrow[d, mapsto] & & \mathcal{E} \arrow[d, "p"] \\
b' \arrow[r, "f = p(\widehat{f})"'] & b & & \mathcal{B}
\end{tikzcd}$$

Fibrations are categories varying over a category, II

Definition

A *fibration* is a functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$ such that

for every e and f there is $\widehat{f}$ universal with the property

$$
\begin{tikzcd}
x \arrow[drr, bend left, "h"] \arrow[dd, mapsto] & & \\
& f^*(e) \arrow[r, "\widehat{f}"'] \arrow[dd, mapsto] & e \arrow[dd, mapsto] \\
p(x) \arrow[dr, "g"'] \arrow[drr, "p(h)"] & & \\
& b' \arrow[r, "f = p(\widehat{f})"'] & b
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{E} \arrow[d, "p"] \\
\mathcal{B}
\end{tikzcd}
$$

Fibrations are categories varying over a category, II

Definition

A *fibration* is a functor $\mathcal{E} \xrightarrow{p} \mathcal{B}$ such that

for every e and f there is $\widehat{f}$ universal with the property

$$
\begin{tikzcd}
x \arrow[dr, dashed, "k"'] \arrow[drr, bend left, "h"] \arrow[dd, mapsto] & & & & \\
& f^*(e) \arrow[r, "\widehat{f}"'] \arrow[dd, mapsto] & e \arrow[dd, mapsto] & & \mathcal{E} \arrow[dd, "p"] \\
p(x) \arrow[dr, "p(k) = g"'] \arrow[drr, bend left, "p(h)"] & & & & \\
& b' \arrow[r, "f = p(\widehat{f})"'] & b & & \mathcal{B}
\end{tikzcd}
$$

Comparing pseudofunctors and fibrations

Pseudofunctors | Fibrations

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{p^{-1}} \mathcal{C}at \longleftarrow\!\!\!\shortmid \; \mathcal{E} \xrightarrow{p} \mathcal{B}$$

the category $p^{-1}(b)$:

$$\begin{tikzcd} e \arrow[d, "v"] & \text{s.t. } p(e) = b \\ e' & \text{s.t. } p(d) = b \end{tikzcd}$$

$v \colon e \to d$ in $\mathcal{E}$
s.t. $p(v) = \mathrm{id}_b$

Comparing pseudofunctors and fibrations

Pseudofunctors

Fibrations

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{p^{-1}} \mathcal{C}at \longleftarrow\!\!\shortmid \mathcal{E} \xrightarrow{p} \mathcal{B}$$

$$\begin{array}{rcl} \text{the functor } p^{-1}(b) & \xrightarrow{p^{-1}(f\colon b' \to b)} & p^{-1}(b') \\ e & \longmapsto & f^*(e) \end{array}$$

$$\begin{tikzcd} f^*(e) \arrow[r, "\widehat{f}"'] \arrow[d, mapsto] & e \arrow[d, mapsto] \\ b' \arrow[r, "f = p(\widehat{f})"'] & b \end{tikzcd}$$

Comparing pseudofunctors and fibrations

Pseudofunctors | Fibrations

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{p^{-1}} \mathcal{C}at \longleftarrow\!\shortmid \mathcal{E} \xrightarrow{p} \mathcal{B}$$

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{C}at \longmapsto \textstyle\int P \xrightarrow{\mathrm{pr}_1} \mathcal{B}$$

the category $\int P$:

$$\underset{\text{s.t. } \alpha' \in P(b')}{(b', \alpha')} \xrightarrow[\substack{f: b' \to b \text{ in } \mathcal{B} \\ h: \alpha \to P(f)(\alpha) \text{ in } P(b')}]{(f, h)} \underset{\text{s.t. } \alpha \in P(b)}{(b, \alpha)}$$

composition is

$$\begin{tikzcd} (b'', \alpha') \arrow[r, "{(g, k)}"] \arrow[rr, bend right, "{(f \circ g, \mu_{g,f} \circ (P(g)(h)) \circ k)}"'] & (b', \alpha') \arrow[r, "{(f, h)}"] & (b, \alpha) \end{tikzcd}$$

A picture of composition in the category $\int P$

$$
\begin{tikzcd}
\alpha'' \arrow[d, "k"] & & \\
P(g)(\alpha') & \alpha' \arrow[l, mapsto, "P(g)"'] \arrow[d, "h"] & \\
& P(f)(\alpha) & \alpha \arrow[l, mapsto, "P(f)"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
P(b'') & P(b') \arrow[l, "P(g)"'] & P(b) \arrow[l, "P(f)"']
\end{tikzcd}
$$

A picture of composition in the category $\int P$

$$
\begin{tikzcd}
\alpha'' \arrow[d, "k"] \arrow[ddd, bend right] & & \\
P(g)(\alpha') & \alpha' \arrow[l, mapsto, "P(g)"'] \arrow[d, "h"] & \\
 & P(f)(\alpha) & \alpha \arrow[l, mapsto, "P(f)"'] \arrow[dll, mapsto, bend left, "P(f \circ g)"] \\
P(f \circ g)(\alpha) & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
P(b'') & P(b') \arrow[l, "P(g)"'] & P(b) \arrow[l, "P(f)"'] \arrow[ll, bend left, "P(f \circ g)"]
\end{tikzcd}
$$

A picture of composition in the category $\int P$

$$
\begin{tikzcd}
\alpha'' \arrow[d, "k"] \arrow[ddd, bend right] & & \\
P(g)(\alpha') \arrow[d, "P(g)(h)"] & \alpha' \arrow[l, mapsto, "P(g)"'] \arrow[d, "h"] & \\
P(g)(P(f)(\alpha)) & P(f)(\alpha) \arrow[l, mapsto, "P(g)"'] & \alpha \arrow[l, mapsto, "P(f)"'] \arrow[dll, mapsto, "P(f \circ g)"] \\
P(f \circ g)(\alpha) & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
P(b'') & P(b') \arrow[l, "P(g)"'] & P(b) \arrow[l, "P(f)"'] \arrow[ll, bend left, "P(f \circ g)"]
\end{tikzcd}
$$

A picture of composition in the category $\int P$

$$
\begin{tikzcd}
\alpha'' \arrow[d, "k"] \arrow[ddd, bend right] & & \\
P(g)(\alpha') \arrow[d, "P(g)(h)"] & \alpha' \arrow[l, mapsto, "P(g)"'] \arrow[d, "h"] & \\
P(g)(P(f)(\alpha)) \arrow[d, "\mu_{g,f}"] & P(f)(\alpha) \arrow[l, mapsto, "P(g)"'] & \alpha \arrow[l, mapsto, "P(f)"'] \arrow[dll, mapsto, bend left, "P(f \circ g)"] \\
P(f \circ g)(\alpha) & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
P(b'') \arrow[r, phantom, "\Downarrow \mu_{g,f}" below] & P(b') \arrow[l, "P(g)"'] & P(b) \arrow[l, "P(f)"'] \arrow[ll, bend left, "P(f \circ g)"]
\end{tikzcd}
$$

Comparing pseudofunctors and fibrations, II

Examples

Pseudofunctor $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{C}at$

$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\wp} \mathcal{P}os \hookrightarrow \mathcal{C}at$

$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}} \mathcal{P}reord \hookrightarrow \mathcal{C}at$

for $\mathcal{T}$ a theory in first order logic

$\mathcal{G}^{\mathrm{op}} \xrightarrow{\xi} \mathrm{Aut}(\mathcal{H}) \hookrightarrow \mathcal{C}at$

$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\mathcal{C}^{(-)}} \mathcal{C}at$

$\mathcal{B}^{\mathrm{op}} \xrightarrow{F} \mathcal{S}et \hookrightarrow \mathcal{C}at$

$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}/(-)} \mathcal{C}at$

$\mathcal{B}$ with pullbacks

Fibration $\int P \xrightarrow{\mathrm{pr}_1} \mathcal{B}$

$\mathcal{S}ub \longrightarrow \mathcal{S}et$

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \cap & & \cap \\ S & \xrightarrow{f} & S' \end{array} \longmapsto S \xrightarrow{f} S'$$

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$\mathcal{B}$ with pullbacks

Fibration $\int P \xrightarrow{\mathrm{pr}_1} \mathcal{B}$

$$(\vec{x}, \varphi) \xrightarrow{\vec{t}} (\vec{x}', \psi) \quad \text{s.t. } \vec{x} | \varphi \vdash_{\mathcal{T}} \psi[\vec{t}/\vec{x}'] \qquad \longmapsto \qquad \vec{x} \xrightarrow{\vec{t}} \vec{x}'$$

Comparing pseudofunctors and fibrations, II

Examples

Pseudofunctor $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{C}at$

Fibration $\int P \xrightarrow{\mathrm{pr}_1} \mathcal{B}$

$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\wp} \mathcal{P}os \hookrightarrow \mathcal{C}at$

$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}} \mathcal{P}reord \hookrightarrow \mathcal{C}at$

for $\mathcal{T}$ a theory in first order logic

$\mathcal{G}^{\mathrm{op}} \xrightarrow{\xi} \mathrm{Aut}(\mathcal{H}) \hookrightarrow \mathcal{C}at$

$\mathcal{G} \ltimes_{\xi} \mathcal{H} \xrightarrow{\mathrm{pr}_1} \mathcal{G}$

$(g, h) \cdot (g', h') = (gg', \xi_{g'}(h)h')$

$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\mathcal{C}^{(-)}} \mathcal{C}at$

$\mathcal{B}^{\mathrm{op}} \xrightarrow{F} \mathcal{S}et \hookrightarrow \mathcal{C}at$

$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}/(-)} \mathcal{C}at$

$\mathcal{B}$ with pullbacks

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$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\mathcal{C}^{(-)}} \mathcal{C}at$

$\mathrm{Fam}(\mathcal{C}) \longrightarrow \mathcal{S}et$

$$(c_i)_{i \in S} \xrightarrow[f_i : c_i \to c'_{r(i)},\, i \in S]{(r, (f_i)_{i \in S})} (c'_j)_{j \in S'} \quad \longmapsto \quad S \xrightarrow{f} S'$$

$\mathcal{B}^{\mathrm{op}} \xrightarrow{F} \mathcal{S}et \hookrightarrow \mathcal{C}at$

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$\int F \xrightarrow{\mathrm{pr}_1} \mathcal{B}$ is a *discrete* fibration

$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}/(-)} \mathcal{C}at$

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Comparing pseudofunctors and fibrations, II

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$\mathcal{B}$ with pullbacks

$\mathcal{B}^2 \xrightarrow{\mathrm{cod}} \mathcal{B}$

$2 = \boxed{\bullet \longrightarrow *}$

Split fibrations

Definition

A fibration $\mathcal{D} \xrightarrow{S} \mathcal{B}$ is *split* when it is of the form $\int P \xrightarrow{\mathrm{pr}_1} \mathcal{B}$ for some functor $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{C}at$

$\mathrm{SplFibr}(\mathcal{B}) = \mathsf{cat}([\mathcal{B}^{\mathrm{op}}, \mathcal{S}et])$

Examples

$\mathcal{S}ub \longrightarrow \mathcal{S}et$

$\mathcal{G} \ltimes_{\xi} \mathcal{H} \xrightarrow{\mathrm{pr}_1} \mathcal{G}$

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Discrete fibrations $\int F \xrightarrow{\mathrm{pr}_1} \mathcal{B}$

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Examples

Representable discrete fibrations

$$\begin{tikzcd} \mathcal{B}/a \arrow[d, "\text{よ}_a"'] \\ \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} & b \arrow[ld, "e"] \\ a & \end{tikzcd} \quad e \mapsto b$$

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Representable discrete fibrations

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Examples

The Yoneda fibration

$$\begin{tikzcd}
\mathcal{B}_1 \arrow[d, "よ"'] \\
\mathcal{B}^{\mathrm{op}} \times \mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & b' \arrow[rr, "f_2"] \arrow[dl, "f_1 e f_2"'] & & b \arrow[dl, "e"] \\
a' \arrow[d, mapsto] & & a \arrow[ll, "f_1"] \arrow[d, mapsto] & \\
(a', b') \arrow[rr, "(f_1, f_2)"'] & & (a, b) &
\end{tikzcd}$$

Split fibrations

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A fibration $\mathcal{D} \xrightarrow{S} \mathcal{B}$ is *split* when it is of the form $\int P \xrightarrow{\mathrm{pr}_1} \mathcal{B}$ for some functor $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{C}at$

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Examples

For C an internal category in $\mathcal{B}$ $\qquad \int_b よ(b, C) \xrightarrow{\mathrm{pr}_1} \mathcal{B}$

$\mathcal{T}op \xrightarrow{U} \mathcal{S}et$

Split fibrations

$$\begin{tikzcd}
 & & \mathcal{C}at/_{\mathcal{B}} \\
\mathrm{SplFibr}(\mathcal{B}) \arrow[urr, hook] \arrow[rr, hook, bend right=0] \arrow[rr, phantom, shift left=2, "\perp"] & & \mathrm{Fibr}(\mathcal{B}) \arrow[u, hook] \arrow[ll, bend right=10]
\end{tikzcd}$$

Split fibrations

$$S(p) = \int_a \mathrm{Fibr}(\mathcal{B})(\text{よ}_a, p) \xrightarrow{\mathrm{pr}_1} \mathcal{B}$$

$$\begin{tikzcd}
 & \mathcal{C}at/_{\mathcal{B}} \\
\mathrm{SplFibr}(\mathcal{B}) \arrow[ur, hook] \arrow[r, hook, "\perp", "\perp"'] & \mathrm{Fibr}(\mathcal{B}) \arrow[u, hook] \arrow[l, bend right] \arrow[l, bend left, "S"]
\end{tikzcd}$$

Theorem (Fibered Yoneda Lemma)

For a fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$ *and a split fibration* $\mathcal{D} \xrightarrow{s} \mathcal{B}$

$$\mathrm{Fibr}(\mathcal{B})(s, p) \equiv \mathrm{SplFibr}(\mathcal{B})(s, S(p))$$

Split fibrations

$$S(p) = \int_a \mathrm{Fibr}(\mathcal{B})(\text{よ}_a, p) \xrightarrow{\mathrm{pr}_1} \mathcal{B}$$

$$R(t) = \int_a \mathcal{C}at/_{\mathcal{B}}(\text{よ}_a, t) \xrightarrow{\mathrm{pr}_1} \mathcal{B}$$

$$\begin{tikzcd}
& & \mathcal{C}at/_{\mathcal{B}} \arrow[dll, bend left=10, "R"] \\
\mathrm{SplFibr}(\mathcal{B}) \arrow[urr, hook, "U"] \arrow[urr, phantom, shift left=2, "\perp"] \arrow[rr, hook] & & \mathrm{Fibr}(\mathcal{B}) \arrow[u, hook] \arrow[ll, bend right=10] \arrow[ll, bend left=10, "S"] \arrow[ll, phantom, shift right=2, "\perp"] \arrow[ll, phantom, shift left=2, "\perp"]
\end{tikzcd}$$

Theorem

For a functor $\mathcal{C} \xrightarrow{t} \mathcal{B}$ *and a split fibration* $\mathcal{D} \xrightarrow{s} \mathcal{B}$

$$\mathcal{C}at/_{\mathcal{B}}(U(s), t) \equiv \mathrm{SplFibr}(\mathcal{B})(s, R(t))$$

Fibrations are pseudo-coalgebras

$$\begin{tikzcd}
\mathcal{C}at/_{\mathcal{B}} \arrow[loop left, "UR"] \arrow[dr, shift left] \arrow[dr, phantom, "\perp"] & \\
\mathrm{Fibr}(\mathcal{B}) \arrow[u, hook] \arrow[r, phantom, "\equiv"] & \text{Ps-}UR\text{-Coalg} \arrow[ul, shift left]
\end{tikzcd}$$

Theorem

The comonad on UR is KZ and the pseudo-coalgebras are fibrations

Fibrations are pseudo-coalgebras

$$
\begin{tikzcd}
\mathcal{C}at/_{\mathcal{B}} \arrow[loop left, "UR"] \arrow[dr, shift left] \arrow[dr, phantom, "\top"] & \\
\mathrm{Fibr}(\mathcal{B}) \arrow[u, hook] \arrow[r, phantom, "\equiv"] & \text{Ps-}UR\text{-Coalg} \arrow[ul, shift left]
\end{tikzcd}
$$

Corollary

For a fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$ *there is a diagram*

$$
\begin{tikzcd}
\mathcal{E} \arrow[loop left, "\mathrm{Id}_{\mathcal{E}}"] \arrow[rr, shift right, "\alpha"'] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & \int_a \mathcal{C}at/_{\mathcal{B}}(\text{よ}_a, p) \arrow[ll, shift right, "y_p"'] \arrow[dl, "R(p)"] \\
& \mathcal{B} &
\end{tikzcd}
$$

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