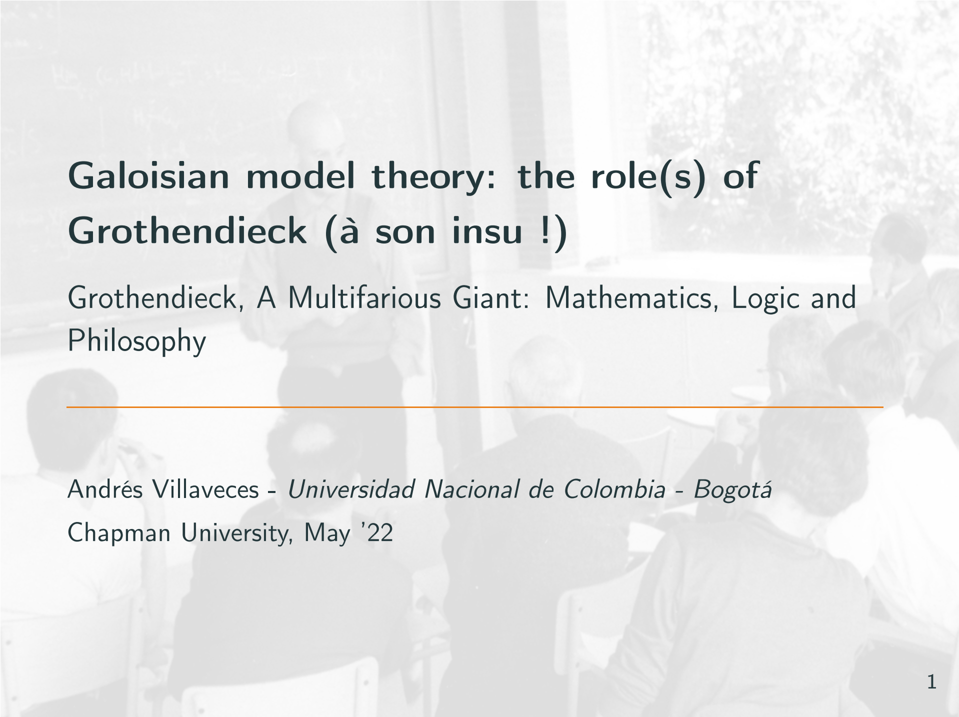


Galoisian model theory: the role(s) of Grothendieck (à son insu !)

Grothendieck, A Multifarious Giant: Mathematics, Logic and Philosophy

Andrés Villaveces - *Universidad Nacional de Colombia - Bogotá*

Chapman University, May ’22

Grothendieck: L'aventure solitaire (R et S, p. 363)

Ce qui m'intéresse dans ce passé, ce n'est nullement ce que j'y ai fait (la fortune qui est ou sera la sienne), mais bien plutôt ce qui n'a pas été fait, dans le vaste programme que j'avais alors devant les yeux, et dont une toute petite partie seulement s'est trouvée réalisée par mes efforts et ceux des amis et élèves qui parfois ont bien voulu se joindre à moi. Sans l'avoir prévu ni cherché, ce programme lui-même s'est renouvelé, en même temps que ma vision et mon approche des choses mathématiques. (...) les **mystères** *qui m'ont le plus fasciné, tel celui des « motifs », ou celui de la description « géométrique » du groupe de Galois de* $\overline{\mathbb{Q}}$ *sur* $\mathbb{Q}$*...*

Our stops today

Galois Theory of Model Theory - First descent

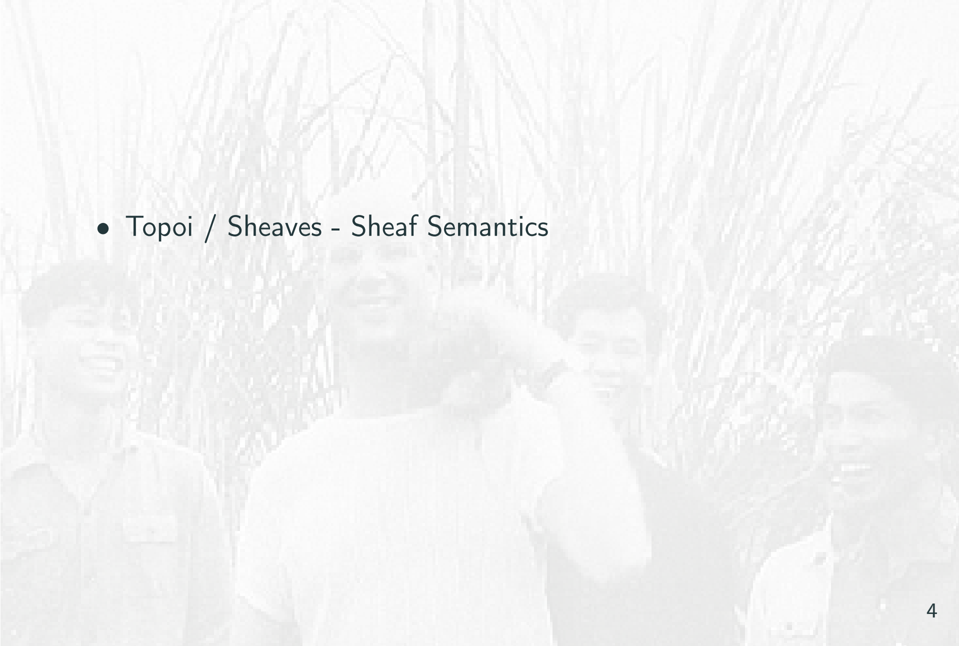
Stability: early link with Grothendieck

- A Grothendieckian variant: Hrushovski-Kamensky

Three Ascents: Hrushovski's Core, Beyond FO, Higher Stability

- Beyond First Order
- Higher Stability?

But . . . Why Grothendieck and Logic?



- Topoi / Sheaves - Sheaf Semantics

But . . . Why Grothendieck and Logic?

- Topoi / Sheaves - Sheaf Semantics
- TODAY: **Galois Theory of Model Theory**

But . . . Why Grothendieck and Logic?

- Topoi / Sheaves - Sheaf Semantics
- TODAY: **Galois Theory of Model Theory**
- (and, as a bonus, a bit about The Role of Stability and his Role in coining out that concept)

Plan

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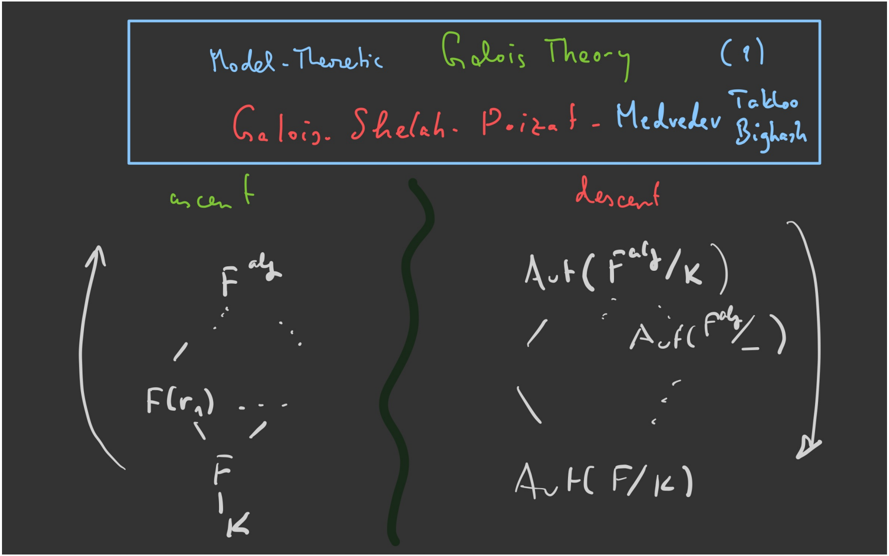
Higher Stability?

Model Theory: a natural Galois-theoretic framework



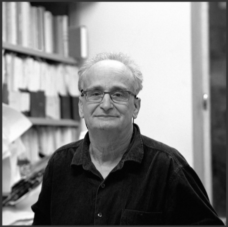
In first order, the key role of imaginaries

~ 1980: Shelah defines imaginaries for ARBITRARY first order theories T:

equivalence classes of definable functions

$$a/_E = \{ b \in M \mid a E^M b \}$$

$$\varphi(a,b) \iff aEb$$



S. Shelah - 2016

Poizat makes the connection explicit

1983 : Poizat notes that Shelah's imaginaries are exactly what's needed for a general Galois theory (for models of T)

Bruno Poizat
(1970s)

THE JOURNAL OF SYMBOLIC LOGIC
Volume 48, Number 4, Dec. 1983

UNE THÉORIE DE GALOIS IMAGINAIRE

BRUNO POIZAT

Introduction. La communauté mathématique doit être reconnaissante à Saharon Shelah pour une invention d'une ingénieuse simplicité, celle d'avoir associé à chaque structure M une structure M^{eq} comprenant, outre les éléments de M, des "éléments imaginaires" qui sont virtuellement présents dans M. La finalité de cette construction est de pourvoir toute formule $f(\bar{x}, \bar{a})$ à paramètres dans M, et même dans M^{eq}, d'un ensemble de définition minimum; tout cela est rappelé dans la

In the stable context, a MUCH CLOSER proof to the original

> Il n'est pas de grande gloire à démontrer un résultat universellement connu depuis 150 ans (qui est exprimé ici de manière anachronique : au temps de Galois, il n'est question ni de corps, ni a fortiori de clôture algébrique) ; cependant cette démonstration a le mérite d'être beaucoup plus proche des preuves, ou des indications de preuve, qu'on trouve dans les manuscrits de Galois, dont elle est en définitive une traduction en langage contemporain ; elle est plus directe que celle qu'on enseigne habituellement dans les cours d'algèbre, qui repose sur l'étude des extensions de corps de degré fini, et qui n'a été mise au point qu'à la fin du siècle dernier ; ceci tout en satisfaisant aux exigences de la sacro-sainte rigueur des mathématiciens d'aujourd'hui (à l'exception de ceux qui ne voient pas en la théorie des modèles une activité mathématique).

Poizat: the model-theoretic proof of Galois' Correspondence is MUCH CLOSER to the (indications of) proof in Galois' manuscripts!

- In what sense?
- With which obstructions?

Some translations (following Medvedev/Takloo-Bighash)

AN INVITATION TO MODEL-THEORETIC GALOIS THEORY.

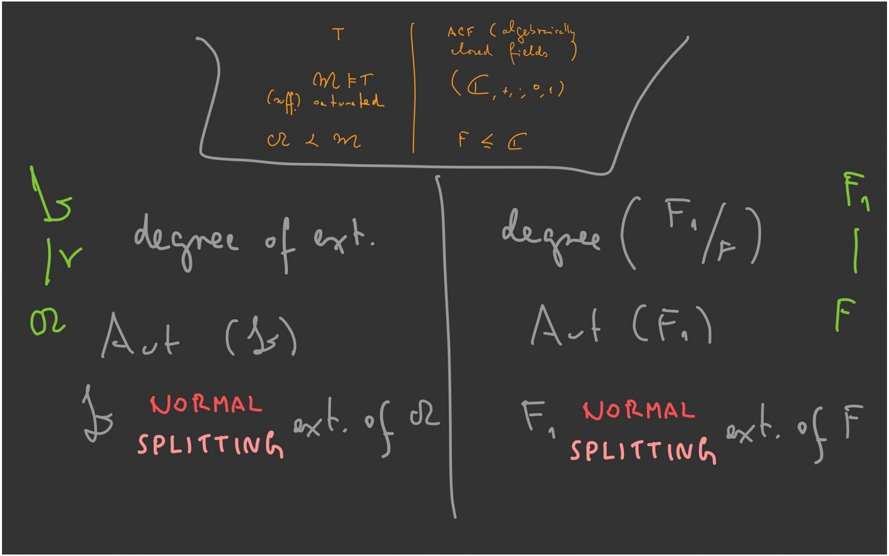
ALICE MEDVEDEV AND RAMIN TAKLOO-BIGHASH

ABSTRACT. We carry out some of Galois' work in the setting of an arbitrary first-order theory T. We replace the ambient algebraically closed field by a large model $\mathcal{M}$ of T, replace fields by definably closed subsets of $\mathcal{M}$, assume that T codes finite sets, and obtain the fundamental duality of Galois theory matching subgroups of the Galois group of L over F with intermediate extensions $F \leq K \leq L$. This exposition of a special case of [11] has the advantage of requiring almost no background beyond familiarity with fields, polynomials, first-order formulae, and automorphisms.

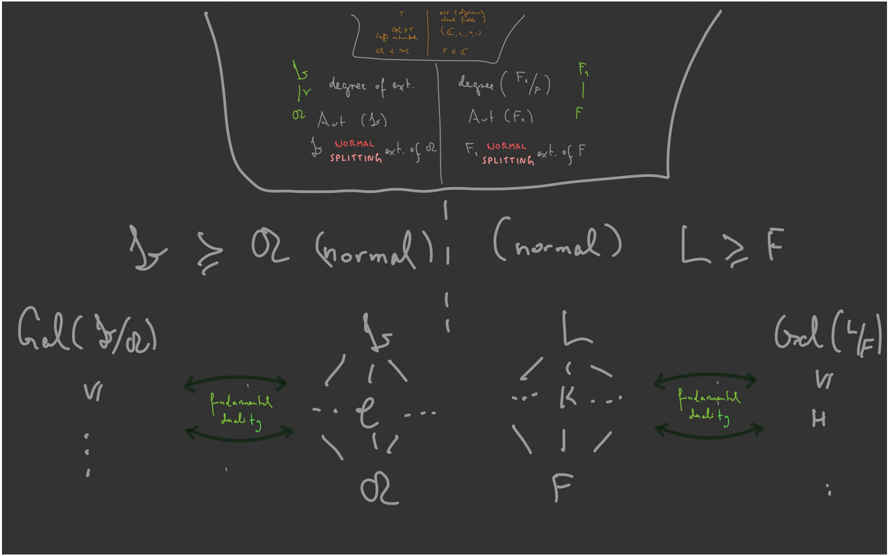
- A. Medvedev, R. Takloo-Bighash: An Invitation to Model-Theoretic Galois Th. 2010

T	ACF (algebraically closed fields)
$\mathcal{M} \models T$ (suff.) saturated	$(\mathbb{C}, +, \cdot, 0, 1)$
$\mathcal{A} \leq \mathcal{M}$	$F \leq \mathbb{C}$

Normal and Splitting Extensions (def. later)



Galois duality I



What is lost?

- L might not have function symbols!
- Polynomials $\rightsquigarrow$ Formulas (QE)
- $L_S > \mathcal{O}$ might be far from linear $/\mathcal{O}$
- the degree of an extension might not be given by vector space DIM
- NO norms, traces, determinants!

A couple of notions for the translation

- $\mathcal{M} \models T$ suff. saturated
- $\varphi(x,y)$ L-fla; $b \in \mathcal{M}$ is a solution of $\varphi(a,y)$ if $\mathcal{M} \models \varphi(a,b)$.
- we have acl, dcl notions:

$$b \in \operatorname{acl}(A) \text{ if } \operatorname{orb}(b/A) \text{ is finite}$$

- … degree of b/A = $|\operatorname{orb}(b/A)|$

$$\begin{array}{c} M \\ | \\ K \\ | \\ Q \end{array}$$

- K is a finite extension of Q if $\exists\, b$ tuple of $K \cap \mathrm{acl}(Q)$ s.t. $K \subseteq \mathrm{dcl}(Q \cup b)$
- K is NORMAL $/Q$ if $\mathrm{orb}(c/Q) \subseteq K$, $\forall c \in K$

Splitting extensions

- $\mathfrak{F}$ is SPLITTING over $\mathrm{irr}(b/A)$, over $\mathfrak{a}$

  if $\begin{cases} \mathrm{orb}(b/\mathfrak{a}) \subseteq \mathfrak{B} \\ \mathfrak{B} \subseteq \mathrm{dcl}(\mathfrak{a} \cup \mathrm{orb}(b/\mathfrak{a})) \end{cases}$

- $\mathfrak{F}$ def. closed $\Rightarrow$ $\mathfrak{F}$ is normal $/\mathfrak{a}$, splits $/\mathfrak{a}$
- $\mathfrak{a} \hookrightarrow \mathfrak{F} \hookrightarrow \mathfrak{C}$ fin.

  $\Rightarrow$ $\deg(\mathfrak{C}/\mathfrak{a}) = \deg(\mathfrak{C}/\mathfrak{F}) \cdot \deg(\mathfrak{F}/\mathfrak{a})$

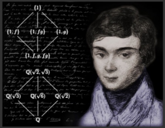
A key step: codifying finite sets

... If T codifies finite sets

$\mathcal{C} = \mathrm{dcl}(\mathcal{C})$ is a normal ext. of $\mathcal{A} = \mathrm{dcl}(\mathcal{A})$

$G = \mathrm{Aut}(\mathcal{C}/\mathcal{A})$

$\Rightarrow$ Subgr. of G — Definably closed intermediate extensions



$G \geq H$ $\longleftrightarrow$ Gal $\longleftrightarrow$ $\mathcal{A} \subseteq \mathrm{Fix}(H) = \{c \in \mathcal{C} \mid \forall h \in H \; h(c) = c\}$

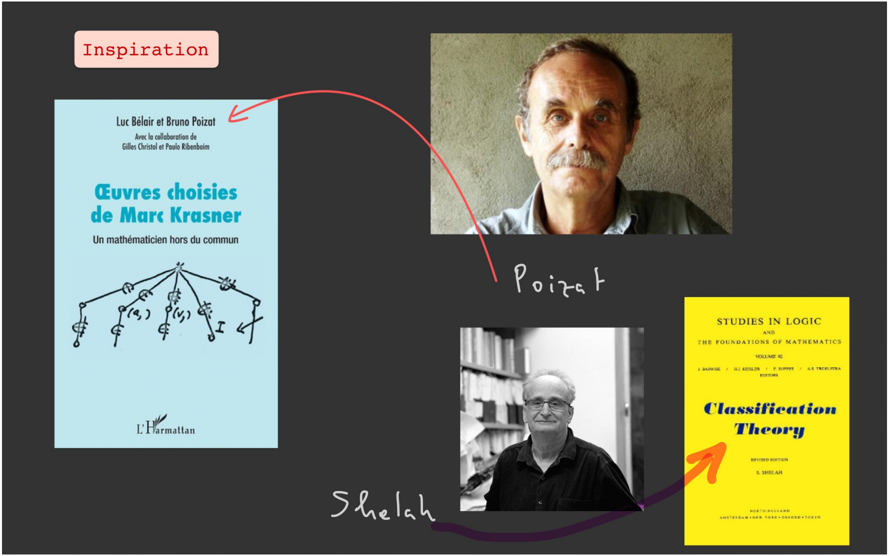
The notion

T codifies finite SETS of tuples i.e.

$$\forall n \in \mathbb{N} \quad \forall F \subset M^n \text{ finite} \quad \exists b \left(\forall \sigma \in Aut(\mathcal{M}) \quad \sigma(b) = b \Leftrightarrow \sigma(F) = F \right)$$

alike SYMMETRIC polynomials

Summary of the first rapprochement: the two sources



A crucial early hypothesis: stability

The key hypothesis to the early possibility of defining a good Galois group of a theory was stability: roughly, a solid theory of definability of orbits (Galois-types) of the action of the automorphism group of a large structure $M \models T$. We now take a détour.

Plan

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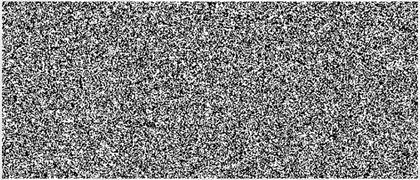
The context for classification



Consider the "formless magma" of all possible mathematical structures. Random noise?

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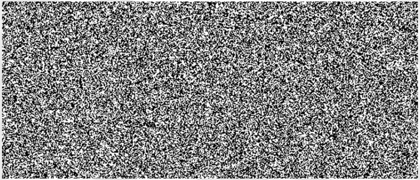
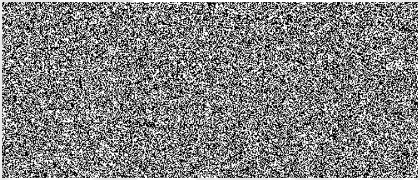


Consider the “formless magma” of all possible mathematical structures. Random noise?

Perhaps not quite! Groups, fields, algebraic varieties, Sobolev spaces, inner models of set theory, function spaces, you name them...

Is it a homogeneous world? What kind of classification is there?

The context for classification



Consider the "formless magma" of all possible mathematical structures. Random noise?

Perhaps not quite! Groups, fields, algebraic varieties, Sobolev spaces, inner models of set theory, function spaces, you name them...

Is it a homogeneous world? What kind of classification is there? Model theory has a strong classification of all FIRST ORDER structures.

Model Theory - a theory of invariants?

$\langle \mathbb{N}, +, <, \cdot, 0, 1 \rangle$ - arithmetics

$\langle \mathbb{C}, +, \cdot, 0, 1 \rangle$ - algebraic geometry

$\langle \mathbb{R}, +, <, \cdot, 0, 1 \rangle$ - real alg. geom.

vector spaces (modules, etc.)

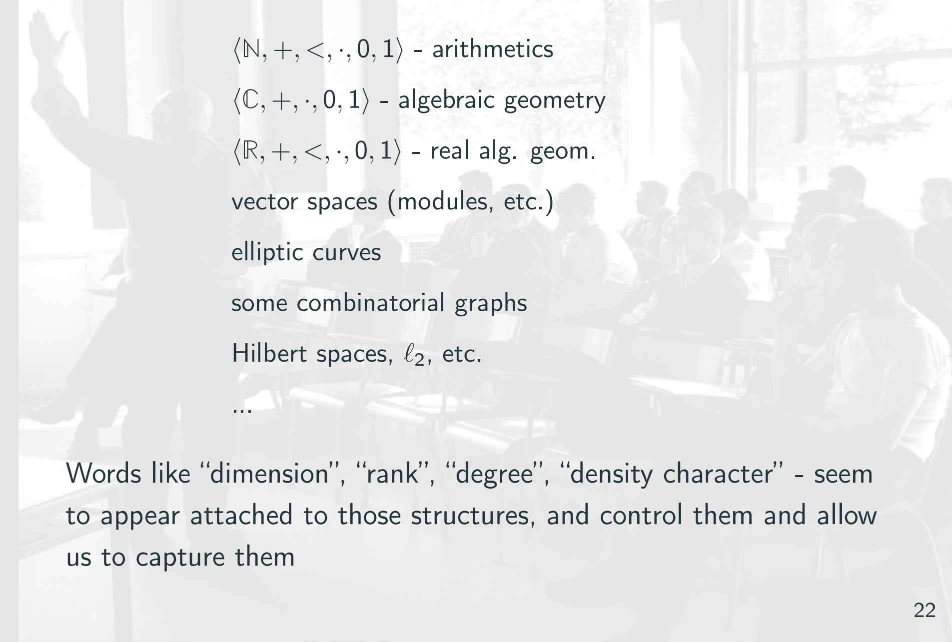
elliptic curves

some combinatorial graphs

Hilbert spaces, ℓ_2, etc.

...

Model Theory - a theory of invariants?



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vector spaces (modules, etc.)

elliptic curves

some combinatorial graphs

Hilbert spaces, ℓ_2, etc.

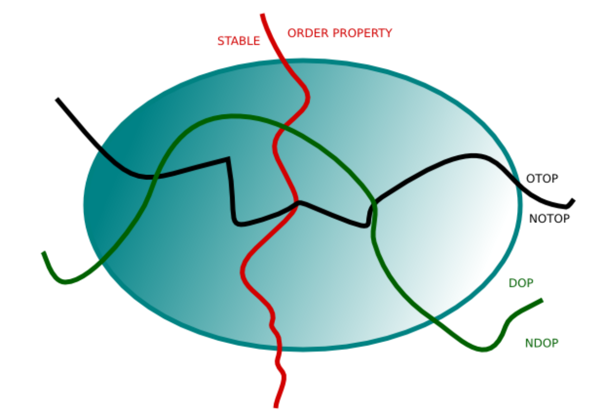
...

Words like "dimension", "rank", "degree", "density character" - seem to appear attached to those structures, and control them and allow us to capture them

Model Theory: perspective and fine-grain

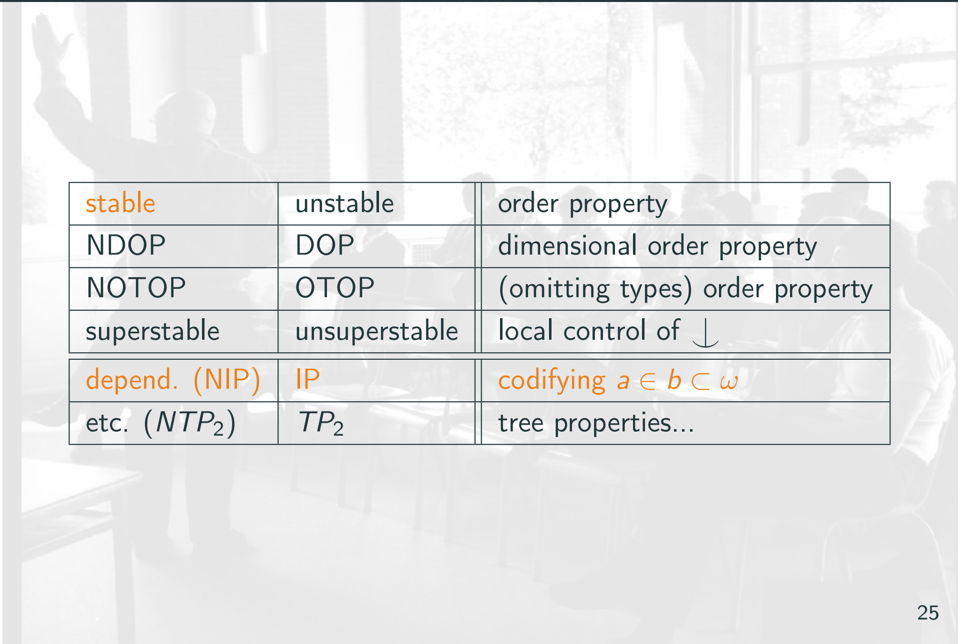
1. Arbitrary **structures**.
2. Hierarchy of types of structures (or their theories): Stability Theory.
3. In the "best part" of the hierarchy: generalized Zariski topology - Zariski Geometries due to Hrushovski and Zilber: algebraic varieties - "arbitrary" structures whose place in the hierarchy ends up automatically giving them strong similarity to elliptic curves.
4. Beyond direct control by a logic: the hierarchy does extend (Abstract Elementary Classes)

Taxonomy



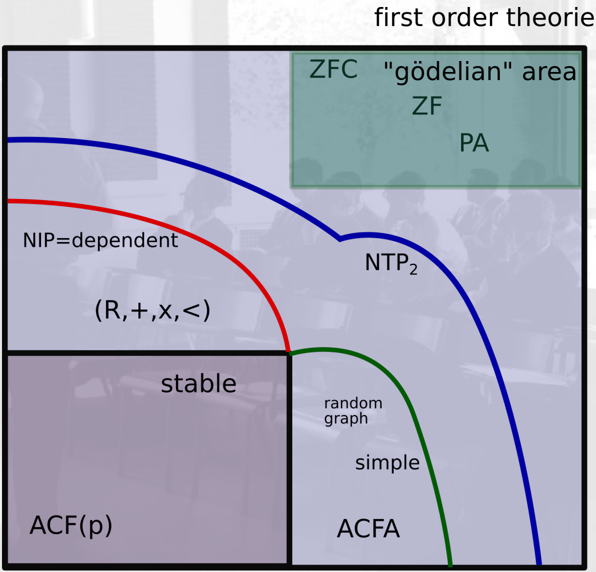
A “taxonomy” of classes of structures.

Dividing Lines

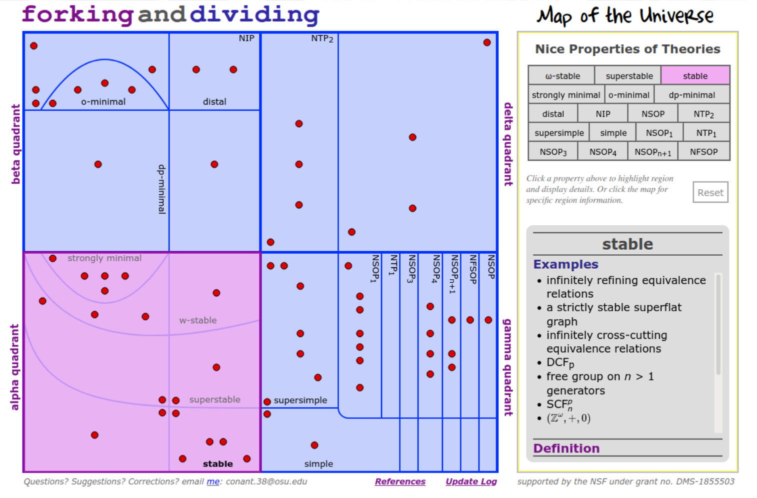


stable	unstable	order property
NDOP	DOP	dimensional order property
NOTOP	OTOP	(omitting types) order property
superstable	unsuperstable	local control of $\mathop{\smile}\limits^{\vert}$
depend. (NIP)	IP	codifying $a \in b \subset \omega$
etc. (NTP_2)	TP_2	tree properties...

A “map” of first order theories



High resolution (forkinganddividing.com / G. Conant)

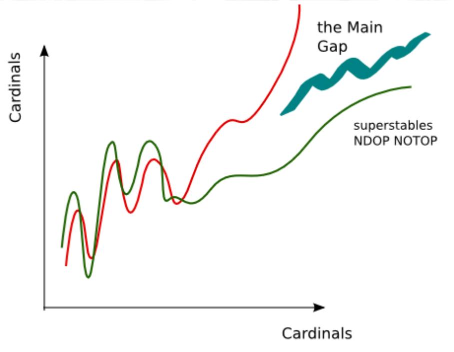


A big aim in Model Theory, triggered by Stability

Given a countable theory T, the spectrum function $I(T,\cdot)$ either always achieves the maximum values, else it has a bound:

$$I(T,\aleph_\alpha) < \beth_{\omega_1}(|\omega+\alpha|)$$

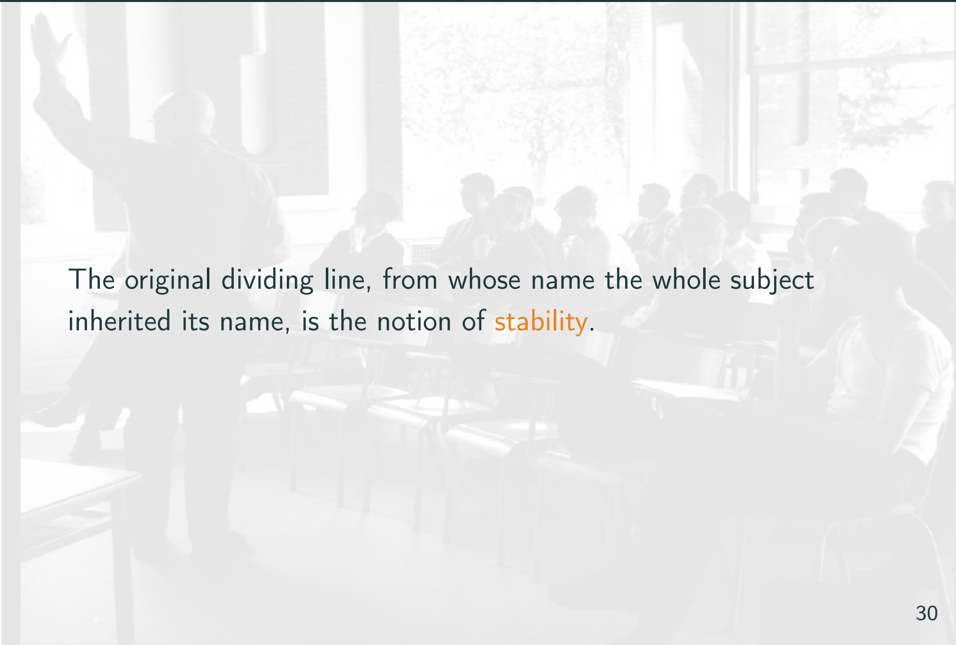
Notice that the result reveals asymptotic behaviour!



Later spin-off from Stability Theory:

- Hrushovski's proof of the Mordell-Lang Conjecture (ca. 1990)
- Model-Theoretic Analysis of the André-Oort Conjecture
- Proof (Casale-Freitag-Nagloo) of a Conjecture by Painlevé from ca. 1895, using the model theory of Differentiably Closed Fields
- Model Theoretic Analysis of analytic functions and Grothendieck's Standard Conjectures (Zilber, since ca. 2000) - this part not only in First Order Model Theory
- ...

The Main Dividing Line: Stability



The original dividing line, from whose name the whole subject inherited its name, is the notion of stability.

Grothendieck, 1952: early insight

Itaï Ben Yaacov has explained how the Fundamental Theorem of Stability (the equivalence between not having an order and definability of types) follows from this theorem due to Grothendieck:

Theorem (Grothendieck, 1952)
Given a topological space X, $X_0 \subseteq X$ a dense subset, then the following are equivalent (for $A \subseteq C_b(X)$, the Banach space of bounded, complex-valued functions on X, equipped with the supremum norm):

- *A is relatively weakly compact in $C_b(X)$,*
- *A is bounded, and for all sequences (f_n), $f_n \in A$ and (x_n), $x_n \in X_0$,*

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m),$$

when both limits exist.

Interpretation between FO theories - Models as functors

(Makkai-Reyes)

- Let us fix a first order theory T in a vocabulary L, and let us consider the category $\mathcal{T}$ of **the definables** of T.

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- Objects are equivalence classes between L-formulas mod T. $A :: \varphi(x)$, etc.
- Morphisms correspond to definable functions: if $A :: \phi(x)$ and $B :: \psi(y)$, a definable morphism $f : A \to B$ is a definable $f :: \chi(x, y)$ such that $T \models \forall x \forall y(\chi(x, y) \to \varphi(x) \wedge \psi(y))$ and $T \models \forall x(\varphi(x) \to \exists y \chi(x, y))$.

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- Given any L-structure $\mathfrak{M}$ and a formula $\varphi(x)$, the **solution set** is $\varphi(\mathfrak{M}) = \{a \in M_x \mid \mathfrak{M} \models \varphi(x)\}$.
- With this, **we regard models of T as functors** from $\mathcal{T}$ to Set: $\mathfrak{M}(A) = \varphi(\mathfrak{M})$. Natural transformations $\equiv$ elementary maps.

Interpretation between FO theories - Models as functors

The category $\mathcal{T} = Def(T)$ is Boolean (regular, and with Boolean algebras of subobjects) and extensive (co-products exist and they form an equivalence between the categories $Sub(X) \times Sub(Y)$ and $Sub(X \sqcup Y)$).

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Boolean categories $\longleftrightarrow$ First Order

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$$\text{Boolean categories} \qquad \longleftrightarrow \qquad \text{First Order}$$

An **interpretation** between T_0 and T is a Boolean and extensive morphism

$$\iota : \mathcal{T}_0 \to \mathcal{T}$$

between the categories $\mathcal{T}_0$ and $\mathcal{T}$ (in the vocabularies L_0 and L).

(ι preserves finite limits, induces homomorphisms of Boolean algebras in subobjects and respects images - and respects co-products)

Interpretation functor between classes of models

We lift the interpretation to classes of models:

Given $\iota : \mathcal{T}_0 \to \mathcal{T}$,

$$\iota^* : Mod(T) \to Mod(T_0)$$

$$\mathfrak{M} \models T \mapsto \iota^*(\mathfrak{M}) = \mathfrak{M}_0$$

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where

$$\mathfrak{M}_0 = \mathfrak{M} \circ \iota : \mathcal{T} \to \mathsf{Set}$$

and if $\sigma : \mathfrak{N} \to \mathfrak{M}$ is an elementary embedding ($\sigma = (\sigma_Y)_{Y \in \mathcal{T}}$) then

$$\iota^*(\sigma) : \mathfrak{N}_0 \to \mathfrak{M}_0 : \iota^* \sigma_X = \sigma_{\iota X}$$

for each $X \in \mathcal{T}_0$.

Examples - ACF, RCF

An interpretation we have known for some 200 years is the following:

$$\iota : Def(ACF) \to Def(RCF)$$

$$\iota(K) = R^2 \text{ , componentwise sum}$$

$$\text{multiplication } (a, b)(\iota\cdot)(c, d) = (ac - bd, bc + ad)$$

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if $R \models RCF$

$$\iota^*(R) = R[\sqrt{-1}].$$

Many other natural examples: retracts, boolean algebras in boolean rings, etc.

Stable Interpretations - a bit on Galois theory

Stability is reflected in a natural way in interpretations:

An interpretation $\iota : \mathcal{T}_0 \to \mathcal{T}$ is **stable** if for each model $\mathfrak{M}$ of T, the "expanded interpretation" $\iota^{\mathfrak{M}} : \mathcal{T}_0^{\mathfrak{M}_0} \to \mathcal{T}^{\mathfrak{M}}$ is an immersion. This means each definable in ιX ($X \in \mathcal{T}_0$) using parameters from $\mathcal{M}$ is the image of a definable set in X using parameters from $\mathcal{M}_0$.

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If T is a stable theory and $\iota : \mathcal{T}_0 \to \mathcal{T}$ is an interpretation, then ι is a stable interpretation and T_0 is a stable theory.

Kamensky, in his thesis (with Hrushovski) went as far as reframing a "Galois theory" of model theory for internal covers - Galois theory à la Grothendieck (SGA 1).

The Galois group of a first order theory

(Assuming that T eliminates imaginaries), A definably closed,

$$Gal(T/A) := Aut(M)/Autf(M)$$

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$$Gal(T/A) := Aut(M)/Autf(M)$$

where M is a saturated model of T and

$$Autf(M) = \langle \bigcup_{A \subset N \prec M} Aut_N(M) \rangle$$

This is an invariant of the theory, allowing a Galois connection between definably closed submodels of M and closed subgroups of the Galois group.

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Galois Theory of Model Theory - First descent

Stability: early link with Grothendieck

A Grothendieckian variant: Hrushovski-Kamensky

Three Ascents: Hrushovski's Core, Beyond FO, Higher Stability

Beyond First Order

Higher Stability?

Ascent 1: Definability patterns (some features)

Around 2019, Hrushovski starts work on "definability patterns" (for first order theories T, or slightly more general contexts).

- Finding the right structure supporting automorphisms allowing a robust Galois correspondence for FO T,

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- Finding models for this theory, and proving their canonicity,
- And going in many directions from here (Galois/Lascar, Ramsey, etc.)

The pattern language: first obstruction

Given $M \models T$, $\mathcal{L}$ consists of predicates Def_t, $t = (\varphi_1, \ldots, \varphi_n; \alpha)$, interpreted in $S = S(M)$ as

$$\mathrm{Def}_t^S = \{(p_1, \ldots p_n) : \forall a \in \alpha(M) \bigvee_{1 \leq i \leq n} (\varphi_i(x, a) \in p_i)\}.$$

For $n = 1$, the predicate $\mathrm{Def}_{\varphi;\alpha}$ captures those 1-types of T for which α acts as a (partial) definition scheme for φ.

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First obstruction beyond First Order: Which formulas to use for definitions???

Possible workarounds

The pattern theory $\mathcal{T}$ of T is the set of all (local) primitive universal $\mathcal{L}$-sentences true in $S(M)$ for some $M \models T$.

Galois-types have very good behaviour in AECs… However, the collection of all Galois-types is not necessarily well-equipped with a "standard" topology!

Definability (of types) has been treated (Shelah, Grossberg, Vasey, VanDieren, Boney, V.) in a weak, abstract way in AECs through non-splitting. Shelah even calls non-splitting extensions in the NIP theories context weakly definable types.

We may use sentences of new logics $\mathbb{L}^{1,\mathrm{aec}}$, to test syntactic definability patterns to build $\mathcal{L}$. Work in progress with my students in Bogotá and with Shelah.

Galois Morleyizations

Vasey in 2016 introduced "Galois Morleyizations" for AECs. Essentially, expanding L by adding predicates for all Galois types (orbits). He proved under "tameness" assumptions that part of the content of an AEC $\mathcal{K}$ may be read functorially from a SYNTACTIC counterpart of the AEC $\mathcal{K}$. In particular, stable AECs have canonical forking relations defined both semantically and syntactically. So far, there is (as far as I have seen) no study of definability of types in that context. But that should enter the picture. . .

Abstract cores (Hrushovski)

A core for T is an $\mathcal{L}$-structure $\mathcal{J}$ such that

- For any (orbit-bounded) $M \models T$, there is an $\mathcal{L}$-embedding

$$j : \mathcal{J} \to S(M)..$$

- For any j as above, there is a retraction $r : S(M) \to \mathcal{J}$ such that $r \circ j = Id_{\mathcal{J}}$.

Cores exist, are unique up to isomorphism. $Aut(\mathcal{J})$ has a natural locally compact topology (basic closed sets of the form $W(R : a, b) = \{g : \mathrm{Def}_t(ga_1, \ldots, ga_n, b_1, \ldots, b_m)\}$

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Calibrating the existence of such cores for additional contexts is doable: choice of logic or plain selection of predicates behaving as if coming from a concrete definability pattern.

The core of j?

Example: the core of j (the j-mapping), as axiomatized by Boris Zilber and Adam Harris in $\mathbb{L}_{\omega_1,\omega}$

$$((\mathbb{H}, \sigma)_{\sigma \in \Gamma}, j, (\mathbb{C}, +, \cdot, 0, 1))$$

is an interesting case for study (here, the quasiminimality of the structure, plus the axiomatization in $\mathbb{L}_{\omega_1,\omega}$ are key).

Higher Stability?

In current work, Chernikov and Towsner have embarked in another theme dear to Grothendieck's ideas: **higher** stability.

Stability may be recast as a question on recovering information about binary relations $R(x, y)$ given with some obstruction, from unary relations $U_1(x)$ and $U_2(y)$. Several classical dividing lines have the format

if $R(x, y)$ satisfies obstruction $(*)$ then, it may be "approximated" by unary relations $U_1(x)$ and $U_2(y)$.

Higher Stability?

In their work (2022), Chernikov and Towsner explore higher-dimensional versions of this phenomenon. There are interesting parallels with Higher Categories.

This is deeply related to much earlier work originally due to Shelah: excellent classes, from the early 1980s. Excellent classes had a major impact on the development of both the deepest theorems in Classification Theory (the so-called Main Gap Theorem) and the understanding of many phenomena in the infinitary logic $L_{\omega_1,\omega}$ and various other non-elementary contexts (AECs).

Thank you all for your attention!

If the Greeks were so attached to geometry, wasn't it that they thought by tracing lines, with no words? However (or maybe just because of that?) [they produced] a perfect axiomatic! Euclid's Postulates, construction. Limiting what one is allowed to trace.

Simone Weil, Cahier III